POLYNOMIAL INVARIANTS FOR $SU(2)$ MONOPOLES

J.M.F. Labastida[★] and M. Mariño[†]

Departamento de Física de Partículas
Universidade de Santiago
E-15706 Santiago de Compostela, Spain

ABSTRACT

We present an explicit expression for the topological invariants associated to $SU(2)$ monopoles in the fundamental representation on spin four-manifolds. The computation of these invariants is based on the analysis of their corresponding topological quantum field theory, and it turns out that they can be expressed in terms of Seiberg-Witten invariants. In this analysis we use recent exact results on the moduli space of vacua of the untwisted $N = 1$ and $N = 2$ supersymmetric counterparts of the topological quantum field theory under consideration, as well as on electric-magnetic duality for $N = 2$ supersymmetric gauge theories.

★ e-mail: labastida@gaes.usc.es
† e-mail: marinho@gaes.usc.es

1. Introduction.

Recently, there has been a great progress in the understanding of the non-perturbative aspects of $N = 1$ [1-5] and $N = 2$ [6, 7] supersymmetric gauge theories in four dimensions. On the one hand, holomorphy constraints and non-perturbative non-renormalization theorems have allowed to obtain exact results for the behavior of the $N = 1$ superpotentials present in a wide class of models. On the other hand, exact results on the quantum moduli space of vacua and on the low-energy effective actions of $N = 2$ supersymmetric Yang-Mills theory and $N = 2$ supersymmetric QCD have been obtained. These achievements have provided an explicit realization of electric-magnetic duality.

One of the most remarkable applications of the exact solution of $N = 2$ pure Yang-Mills theory has been the reformulation in [8] of Donaldson theory [9, 10,11] with gauge group $SU(2)$. It is by now well-known [12] that Donaldson theory can be formulated as a certain twisted version of $N = 2$ Yang-Mills theory. Using electric-magnetic duality of this model one can obtain an equivalent theory which involves an abelian connection coupled to matter in a pair of monopole equations. The new moduli problem is much more tractable than the original one, and it turns out that the Donaldson polynomial invariants [10] can be expressed in terms of certain topological invariants associated to the abelian theory and called Seiberg-Witten invariants. The topological quantum field theory associated to this new moduli space has been constructed in [13] using the Mathai-Quillen formalism. Donaldson invariants for Kähler manifolds were computed previously by Witten in [14]. He showed that on a Kähler manifold it is possible to obtain a topological symmetry for the $N = 2$ Yang Mills theory which comes from an $N = 1$ subalgebra in such a way that the topological character is preserved after perturbing the original theory with an $N = 1$ supersymmetric mass term. The resulting $N = 1$ theory reduces at low energies to the $N = 1$ pure Yang-Mills theory and therefore one can use the results about its vacuum structure [15] to compute the correlation functions. The same procedure has been applied in [16] to compute the partition function of

$N = 4$ Yang-Mills theory on a Kähler manifold.

The monopole equations proposed in [8] have a natural non-abelian generalization which appears in topological quantum field theories involving the minimal coupling of Donaldson-Witten theory to a twisted $N = 2$ matter multiplet. These theories were constructed in [17,18,19], and related, more general topological quantum field theories have been analyzed in [20]. The non-abelian monopole equations were studied in [21] as a generalization of Donaldson theory on four-manifolds, and the corresponding topological quantum field theory was constructed in geometrical terms using the Mathai-Quillen formalism. Other studies of these equations can be found in [20,22]. $U(N)$ monopole equations have been considered from a mathematical point of view in [23, 24], where their relation to vortex equations on Kähler manifolds [25, 26] is stressed.

The aim of the present paper is to compute the topological correlation functions of the topological field theory associated to $SU(2)$ monopoles in the fundamental representation of the gauge group. This gives, as in the Donaldson-Witten case, topological invariants which are polynomials in the two-dimensional and four-dimensional cohomology of the moduli space. The strategy of the computation is the following. First we will show that the topological field theory introduced in [21] is equivalent to a twisted $N = 2$ supersymmetric Yang-Mills theory coupled to one matter hypermultiplet, and that on a Kähler manifold one can obtain a topological symmetry coming from a $N = 1$ subalgebra, extending in this way the result of [14] for pure $N = 2$ supersymmetric Yang-Mills. This makes possible to compute in the $N = 1$ theory obtained after perturbing with a mass term. The vacuum structure of the resulting theory is obtained using the low-energy description of the $N = 2$ theory in [7]. The computation of the polynomial invariants is performed on a Kähler manifold following the procedure in [14]. Then we will use electric-magnetic duality of the $N = 2$ theory and the results of [8] to obtain a general expression for spin manifolds. This expression can be written in terms of Seiberg-Witten invariants, as one should guess from the analysis in [7]. Therefore, Seiberg-Witten invariants seem to underlie the moduli space of anti-self-dual

(ASD) $SU(2)$ instantons as well as the moduli space of $SU(2)$ monopoles.

The paper is organized as follows. In sect. 2 we review the construcion of [21] and we relate it to the standard topological twisting of $N = 2$ supersymmetric QCD. In addition, we present the observables of the theory, we perform the twist on a Kähler manifold, and formulate the perturbed theory. In sect. 3 the vacuum structure of the $N = 2$ and $N = 1$ theory is analyzed and we obtain the symmetry patterns needed in our computations. In sect. 4 we compute the polynomial invariants, first from the $N = 1$ point of view on a Kähler manifold, and then using electric-magnetic duality and the low-energy structure of the $N = 2$ theory on a general spin manifold. In sect. 5 we consider the twisted $N = 2$ supersymmetric QCD theory with a massive hypermultiplet, and we obtain the vacua of the perturbed $N = 1$ supersymmetric theory in order to support the previous analysis. In sect. 6 we state our conclusions and prospects for future work. The first appendix contains some observations about the parity symmetry of the $N = 2$ theory. Finally, in the second appendix we rederive the results about the vacuum structure of the $N = 1$ theory from its exact superpotential.

2. Non-abelian monopoles

In this section we will make a brief presentation of the non-abelian monopole equations and their corresponding topological quantum field theory. We refer the reader to [21] for the details of the construction. In addition, we will discuss how this theory can be regarded as a twisted version of $N = 2$ non-abelian Yang-Mills theory coupled to a massless $N = 2$ matter hypermultiplet. Finally we will consider the theory on a Kähler manifold, and we will show how in this case the theory can be perturbed generating an $N = 1$ supersymmetric mass term while preserving the topological character of the theory.

2.1. NON-ABELIAN MONOPOLE EQUATIONS

Let X be an oriented, compact, spin four-manifold endowed with a Riemannian structure given by a metric g. We will restrict our analysis to spin manifolds since the arguments used in the following sections are only valid for this type of manifolds. The generalization of the non-abelian monopole equations for other manifolds can be done using a Spin_c structure. Work in this direction has appeared recently [23,22]. The positive and negative chirality spin bundles of X will be denoted by S^+ and S^-, respectively. Let P be a principal fibre bundle with some compact, connected, simple group G, and let E be an associated vector bundle to the principal bundle P via a representation R of G. The Lie-algebra associated to G will be denoted by $\mathbf{g}$. Given this data we will consider the field space $\mathcal{M} = \mathcal{A} \times \Gamma(X, S^+ \otimes E)$, where $\mathcal{A}$ is the space of G-connections on E, and $\Gamma(X, S^+ \otimes E)$ is the space of positive-chirality spinors taking values in this representation space. The group $\mathcal{G}$ of gauge transformations of the bundle E acts locally on the elements of $\mathcal{M}$ in the following way:

$$
\begin{aligned}
g^*(A_\mu) &= -igd_\mu g^{-1} + gA_\mu g^{-1}, \\
g^*(M_\alpha) &= gM_\alpha,
\end{aligned}
\tag{2.1}
$$

where $A \in \mathcal{A}$, $M \in \Gamma(X, S^+ \otimes E)$, and g takes values in the representation R of the group G. Notice that in (2.1) while μ is a space-time index, α is a positive-

chirality spinor index. In this paper we use the same notation as in [21] where the index conventions are described in detail. In terms of the covariant derivative $d_A = d + i[A, \ ]$, the infinitesimal form of the transformations (2.1) can be considered as a linear operator:

$$C(\phi) = (-d_A\phi, i\phi M) \in \Omega^1(X, \mathbf{g}_E) \oplus \Gamma(X, S^+ \otimes E), \quad \phi \in \Omega^0(X, \mathbf{g}_E). \quad (2.2)$$

being ϕ such that $g = \exp(i\phi)$. Let us consider a trivial vector bundle $\mathcal{V}$ over $\mathcal{M}$ with fibre $\mathcal{F} = \Omega^{2,+}(X, \mathbf{g}_E) \oplus \Gamma(X, S^- \otimes E)$, where the self-dual differential forms take values in the Lie-algebra representation, $\mathbf{g}_E$, associated to R. The non-abelian monopole equations define a moduli space which is the zero locus of a section on this bundle, $s : \mathcal{M} \longrightarrow \mathcal{V}$. Actually, due to the presence of the gauge symmetry (2.1) one must account for the action of the gauge group $\mathcal{G}$ in both, $\mathcal{M}$ and $\mathcal{V}$. One must therefore consider the associated section $\hat{s} : \mathcal{M}/\mathcal{G} \longrightarrow \mathcal{V}/\mathcal{G}$. The resulting moduli space will be denoted by $\mathcal{M}_{\mathrm{NA}}$.

We will restrict ourselves to the case $G = SU(N)$ and R its fundamental representation, $R = \mathbf{N}$. The generalization of the monopole equations to other simple gauge groups and other representations is straightforward. The non-abelian monopole equations take the form [21]:

$$
\begin{aligned}
&F_{\alpha\beta}^{+ij} + \frac{i}{2}(\overline{M}_{(\alpha}^j M_{\beta)}^i - \frac{\delta^{ij}}{N}\overline{M}_{(\alpha}^k M_{\beta)}^k) = 0, \\
&(D_{\alpha\dot\alpha}M^\alpha)^i = 0,
\end{aligned}
\quad (2.3)
$$

where $F_{\alpha\beta}^{+ij}$ are in the fundamental representation, i.e., $F_{\alpha\beta}^{+ij} = F_{\alpha\beta}^{+a}(T^a)^{ij}$, being T^a, $a = 1, \ldots, N^2 - 1$, the generators of the Lie algebra in the representation $\mathbf{N}$. In the first equation of (2.3) (and similar ones in this paper), a sum in the repeated index k is understood. The second equation in (2.3) is simply the Dirac equation with the Dirac operator coupled to the gauge connection in the fundamental representation.

The section of the bundle $\mathcal{V}$, $s : \mathcal{M} \longrightarrow \mathcal{V}$, corresponding to (2.3) has the following form:

$$s(A, M) = \left(\frac{a}{\sqrt{2}} \left(F_{\alpha\beta}^{+ij} + \frac{i}{2} (\overline{M}_{(\alpha}^j M_{\beta)}^i - \frac{\delta^{ij}}{d_R} \overline{M}_{(\alpha}^k M_{\beta)}^k) \right), (D_{\alpha\dot\alpha} M^\alpha)^i \right), \qquad (2.4)$$

where a is complex number different from zero. This number a was taken to be one in [21] because then certain useful vanishing theorems can be utilized, as first shown in [14] for the abelian case. As it will be dicussed below, the observables of the topological quantum field theory associated to the section (2.4) are independent of the value chosen for a as long as $a \neq 0$.

Some aspects of the moduli space of solutions of the non-abelian monopole equations (2.3) modulo gauge transformations have been studied in [21]. In particular, the virtual dimension of this moduli space, $\dim \mathcal{M}_{\mathrm{NA}}$, turns out to be:

$$\dim \mathcal{M}_{\mathrm{NA}} = \dim \mathcal{M}_{\mathrm{ASD}} + 2 \operatorname{index} D = (4N-2)c_2(E) - \frac{N^2 - 1}{2}(\chi+\sigma) - \frac{d_R}{4}\sigma, \quad (2.5)$$

where χ and σ are the Euler characteristic and the signature, respectively, of the manifold X, and $c_2(E)$ is the second Chern class of the representation bundle and equals the instanton number k. In (2.5) $\mathcal{M}_{\mathrm{ASD}}$ denotes the moduli space of anti-self-dual (ASD) instantons and $\dim \mathcal{M}_{\mathrm{ASD}}$ its virtual dimension,

$$\dim \mathcal{M}_{\mathrm{ASD}} = 4N\, c_2(E) - (N^2 - 1)(\chi + \sigma)/2, \qquad (2.6)$$

which is the index of the ASD complex:

$$0 \longrightarrow \Omega^0(X, \mathbf{g}_E) \xrightarrow{d_A} \Omega^1(X, \mathbf{g}_E) \xrightarrow{p^+ d_A} \Omega^{2,+}(X, \mathbf{g}_E) \longrightarrow 0. \qquad (2.7)$$

index D denotes the index of the Dirac operator coupled to the connection on E , which is given by:

$$\operatorname{index} D = \int_X \mathrm{ch}(E)\hat{A}(X) = -\frac{N}{8}\sigma - c_2(E). \qquad (2.8)$$

Notice that on a four-dimensional spin manifold the index of the Dirac complex is given by $-\sigma/8$, and is always an integer. Therefore $\sigma \equiv 0 \mod 8$. Also notice

6

that on a four-dimensional Kähler manifold

$$\chi + \sigma = 2 - 2b_1 + b_2^+ = 4(1 - h^{1,0} + h^{2,0}), \qquad (2.9)$$

where b_1 is the first Betti number, b_2^+ is the dimension of $H^{2,+}(X)$, and $h^{1,0}$, $h^{2,0}$ denote Hodge numbers. Therefore, on a Kähler manifold, the quantity

$$\Delta = \frac{\chi + \sigma}{4}, \qquad (2.10)$$

is always an integer.

When dealing with moduli spaces associated to the solutions of certain equations, as it happens in our case, one must require certain conditions in order to have a well defined moduli problem. These requirements concern the orientability of the moduli space (which is equivalent to require that the corresponding topological field theory does not have global anomalies) and the free action of the group of gauge transformations on the space of solutions. As it was argued in [21], these conditions are fulfilled in the non-abelian monopole problem as long as they are fulfilled in the Donaldson theory with the same gauge group. In our case we are concerned with $SU(2)$, and the corresponding conditions reduce to $b_2^+ > 1$. In the following we will suppose that this condition holds for our four-dimensional spin manifold X. On a Kähler manifold, $b_2^+ = 2h^{2,0} + 1$ and the above condition is equivalent to $H^{2,0}(X) \neq 0$.

In [21] a preliminary analysis of the moduli space of solutions of the $SU(N)$ monopole equations on compact Kähler manifolds was done. This moduli space has three branches: one of them corresponds to $M = 0$ and is the moduli space of ASD instantons of Donaldson theory, which is thus contained in $\dim \mathcal{M}_{\mathrm{NA}}$. The second branch corresponds to pairs consisting of an equivalence class of holomorphic $Sl(N, \mathbf{C})$ bundles together with a holomorphic section of $K^{1/2} \otimes E$ modulo $Sl(N, \mathbf{C})$ gauge transformations. The third branch is similar to the second branch, but we must consider instead holomorphic sections of $K^{1/2} \otimes \overline{E}$. In addition we

7

need some stability conditions for the pair in order to guarantee the existence of solutions. These stability conditions have an algebraico-geometric character and appear in the Hermite-Einstein equations [27], in the Hitchin equations on Riemann surfaces and in the vortex equations [25, 26]. The non-abelian monopole equations on a Kähler manifold with gauge group $SU(N)$ are closely related to vortex equations where, as we are taking the tensor product of the original bundle E or its conjugate with $K^{1/2}$, the resulting bundle has a fixed determinant. This situation has been analyzed in [24] where the corresponding stability condition has been obtained. Further work on the relation between non-abelian monopole equations and the vortex equations will appear elsewhere [29].

The topological action corresponding to the moduli problem leading to $\mathcal{M}_{\mathrm{NA}}$ was constructed in [21] using the Mathai-Quillen formalism. In order to present the form of this action we need to introduce first a variety of fields. Let (ψ, μ) be an element of the tangent space to the moduli space $\mathcal{M}$ at the point (A, M),
$$(\psi, \mu) \in T_{(A,M)}\mathcal{M} = T_A\mathcal{A} \oplus T_M\Gamma(X, S^+ \otimes E) = \Omega^1(X, \mathbf{g}_E) \oplus \Gamma(X, S^+ \otimes E),$$
and let ϕ be an element of $\Omega^0(X, \mathbf{g}_E)$, $\phi \in \Omega^0(X, \mathbf{g}_E)$. The fields (A, M), (ψ, μ) and ϕ have ghost numbers 0, 1 and 2, respectively. Associated to the fiber $\mathcal{F}$ we introduce fields $(\chi, v_{\dot\alpha}) \in \mathcal{F} = \Omega^{2,+}(X, \mathbf{g}_E) \oplus \Gamma(X, S^- \otimes E)$, with ghost number -1. In addition, fields λ and η in $\Omega^0(X, \mathbf{g}_E)$ with ghost number -2, and -1, respectively, as well as an auxiliary sector made out of ghost-numer zero fields $(H, h) \in \Omega^{2,+}(X, \mathbf{g}_E) \oplus \Gamma(X, S^- \otimes E)$ are introduced. All fields with even ghost number are commuting while the ones with odd ghost number are anticommuting.

The topological action can be written very simply with the help of the BRST symmetry present in the formalism. Under this symmetry the fields of the theory transform in the following way:

$$[Q, A] = \psi,$$
$$\{Q, \psi\} = d_A\phi,$$
$$[Q, \phi] = 0,$$
$$\{Q, \chi_{\mu\nu}\} = H_{\mu\nu},$$
$$[Q, H_{\mu\nu}] = i[\chi_{\mu\nu}, \phi],$$
$$[Q, \lambda] = \eta,$$

$$[Q, M^i_\alpha] = \mu^i_\alpha,$$
$$\{Q, \mu^i_\alpha\} = -i\phi^{ij} M^j_\alpha,$$
$$\{Q, v^i_{\dot\alpha}\} = h^i_{\dot\alpha},$$
$$[Q, h^i_{\dot\alpha}] = -i\phi^{ij} v^j_{\dot\alpha},$$
$$\{Q, \eta\} = i[\lambda, \phi]. \tag{2.11}$$

This symmetry closes up to a gauge transformation (2.1) generated by the field $-\phi$. As anticipated, the action is written as a Q-exact expression:

$$
\begin{aligned}
S = \Big\{ Q, \int_X e \Big[& i\chi^{\alpha\beta ji} \Big(\frac{a}{\sqrt{2}} \big(F^{+ij}_{\alpha\beta} + \frac{i}{2}(\overline{M}^j_{(\alpha} M^i_{\beta)} - \frac{\delta^{ij}}{d_R} \overline{M}^k_{(\alpha} M^k_{\beta)}) \big) - \frac{i}{4} H_{\alpha\beta} \Big) \\
& + \frac{i}{2}(\bar{v}^{\dot\alpha} D_{\alpha\dot\alpha} M^\alpha + \overline{M}^\alpha D_{\alpha\dot\alpha} v^{\dot\alpha}) + \frac{1}{8}(\bar{v}^{\dot\alpha} h_{\dot\alpha} - \bar{h}_{\dot\alpha} v^{\dot\alpha}) \\
& + i\mathrm{Tr}(\lambda \wedge *d^*_A \psi) + \frac{1}{2}(\bar\mu^\alpha \lambda M_\alpha - \overline{M}^\alpha \lambda \mu_\alpha) \Big] \Big\} = S_0 + S_M
\end{aligned}
\tag{2.12}
$$

where,

$$
\begin{aligned}
S_0 = \int_X \Big\{ e \Big[& \frac{ia}{\sqrt{2}} H^{\alpha\beta ji} F^{+ij}_{\alpha\beta} - \frac{ia}{\sqrt{2}} \mathrm{Tr}\big(\chi^{\alpha\beta}(p^+(d_A\psi))_{\alpha\beta} \big) \\
& + \frac{1}{4}\mathrm{Tr}(H^{\alpha\beta} H_{\alpha\beta}) - \frac{i}{4}\mathrm{Tr}(\chi^{\alpha\beta}[\chi_{\alpha\beta}, \phi]) \Big] \\
& - \mathrm{Tr}\big(-i\eta \wedge *d^*_A\psi - i\lambda \wedge *d^*_A d_A\phi - \lambda \wedge *[*\psi, \psi] \big) \Big\}
\end{aligned}
\tag{2.13}
$$

and,

$$
\begin{aligned}
S_M = \int_X e \Big[& \frac{-a}{2\sqrt{2}} H^{\alpha\beta ji} (\overline{M}^j_{(\alpha} M^i_{\beta)} - \frac{\delta^{ij}}{N} \overline{M}^k_{(\alpha} M^k_{\beta)}) \\
& - \frac{a}{\sqrt{2}}(\bar\mu_\alpha \chi^{\alpha\beta} M_\beta - \overline{M}_\alpha \chi^{\alpha\beta} \mu_\beta) + \frac{i}{2}(\bar{h}^{\dot\alpha} D_{\alpha\dot\alpha} M^\alpha + \overline{M}^\alpha D_{\alpha\dot\alpha} h^{\dot\alpha}) \\
& - \frac{i}{2}(\bar{v}^{\dot\alpha} D_{\alpha\dot\alpha} \mu^\alpha - \bar\mu^\alpha D_{\alpha\dot\alpha} v^{\dot\alpha}) - \frac{1}{2}(\overline{M}^\alpha \psi_{\alpha\dot\alpha} v^{\dot\alpha} - \bar{v}^{\dot\alpha} \psi_{\alpha\dot\alpha} M^\alpha) \\
& - \frac{1}{2}(\bar\mu^\alpha \eta M_\alpha + \overline{M}^\alpha \eta \mu_\alpha) - (\bar\mu^\alpha \lambda \mu_\alpha - \frac{i}{2}\overline{M}^\alpha \{\phi, \lambda\} M_\alpha) \\
& + \frac{1}{4}(\bar{h}^{\dot\alpha} h_{\dot\alpha} + i\bar{v}^{\dot\alpha} \phi v_{\dot\alpha}) \Big].
\end{aligned}
\tag{2.14}
$$

9

The action S_0 is the one corresponding to Donaldson-Witten theory. Its observables lead to standard Donaldson invariants. The action S_M contains the 'matter fields' and their couplings to Donaldson-Witten theory. The observables associated to the total action S lead to new topological invariants. Notice that the coefficient a enters in (2.12) multiplying a Q-exact term. This means that any variation in a is Q-exact and therefore, using standard arguments [12], the observables of the theory are independent of a as long as $a \neq 0$. In the rest of this subsection we will take $a = 1$.

The action S_0 differs from the one in [12] by a term of the form:

$$\left\{ Q, \int_X e\,\text{Tr}(\lambda[\eta, \phi]) \right\}. \tag{2.15}$$

This term appears naturally when Donaldson-Witten theory is regarded as a twist of $N = 2$ supersymmetry, but its presence is rather unnatural from the point of view of the Mathai-Quillen formalism. However, being a Q-exact term involving products of fields, the observables of the theory do not depend on it.

After integrating out the auxiliary fields $H_{\alpha\beta}$ and $h_{\dot\alpha}$ in the action (2.12) one finds the on-shell action given in [21] (recall that we set $a = 1$):

$$
\begin{aligned}
\tilde{S} = &\int_X e\Big[g^{\mu\nu}D_\mu\overline{M}^\alpha D_\nu M_\alpha + \frac{1}{4}R\overline{M}^\alpha M_\alpha + \frac{1}{2}\text{Tr}(F^{+\alpha\beta}F^+_{\alpha\beta}) - \frac{1}{8}(\overline{M}^{(\alpha}T^a M^{\beta)})(\overline{M}_{(\alpha}T^a M_{\beta)})\Big] \\
&+ \int_X \text{Tr}\Big(\eta \wedge *d_A^* \psi - \frac{i}{\sqrt{2}}\chi^{\alpha\beta}(p^+(d_A\psi))_{\alpha\beta} - \frac{i}{4}\chi^{\alpha\beta}[\chi_{\alpha\beta}, \phi] + i\lambda \wedge *d_A^* d_A\phi + \lambda \wedge *[*\psi, \psi]\Big) \\
&+ \int_X e\Big(-i\overline{M}^\alpha\{\phi, \lambda\}M_\alpha + \frac{1}{\sqrt{2}}(\overline{M}_\alpha\chi^{\alpha\beta}\mu_\beta - \bar\mu_\alpha\chi^{\alpha\beta}M_\beta) - \frac{i}{2}(\bar v^{\dot\alpha}D_{\alpha\dot\alpha}\mu^\alpha - \bar\mu^\alpha D_{\alpha\dot\alpha}v^{\dot\alpha}) \\
&\qquad - \frac{1}{2}(\overline{M}^\alpha\psi_{\alpha\dot\alpha}v^{\dot\alpha} - \bar v^{\dot\alpha}\psi_{\alpha\dot\alpha}M^\alpha) - \frac{1}{2}(\bar\mu^\alpha\eta M_\alpha + \overline{M}^\alpha\eta\mu_\alpha) + \frac{i}{4}\bar v^{\dot\alpha}\phi v_{\dot\alpha} - \bar\mu^\alpha\lambda\mu_\alpha\Big). \\
&\tag{2.16}
\end{aligned}
$$

2.2. Observables

The observables of the theory are those operators in the cohomology of Q. From the transformations (2.11) follow that the observables in the ordinary $SU(2)$ Donaldson-Witten theory are also observables in this theory. These observables are based on the k-forms operators $\mathcal{O}^{(k)}$ in [12]:

$$
\begin{aligned}
\mathcal{O}^{(0)} &= -\frac{1}{4}\mathrm{Tr}(\phi^2), \\
\mathcal{O}^{(1)} &= -\frac{1}{2}\mathrm{Tr}(\phi\psi), \\
\mathcal{O}^{(2)} &= \frac{1}{2}\mathrm{Tr}(i\phi F - \frac{1}{2}\psi \wedge \psi),
\end{aligned}
\qquad
\begin{aligned}
\mathcal{O}^{(3)} &= \frac{i}{2}\mathrm{Tr}(\psi \wedge F), \\
\mathcal{O}^{(4)} &= \frac{1}{2}\mathrm{Tr}(F \wedge F).
\end{aligned}
\tag{2.17}
$$

These operators satisfy the descent equations,

$$
d\mathcal{O}^{(k)} = \{Q, \mathcal{O}^{(k+1)}\}.
\tag{2.18}
$$

From these equations follow that if Σ is a k-dimensional homology cycle, then

$$
I(\Sigma) = \int_\Sigma \mathcal{O}^{(k)},
\tag{2.19}
$$

is in the cohomology of Q. For simply connected four-manifolds, which is the case of interest in this paper, k-dimensional homology cycles only exist for $k = 0, 2, 4$. For $k = 4$ the cycle Σ is the four-manifold X and $I(X)$ is the instanton number.

We have not found any new invariant involving matter fields. This problem should be addressed from the point of view of the universal instanton, but presumably the absence of matter invariants means that the universal bundle associated to the non-abelian monopole equations is the pullback of the universal bundle associated to Donaldson theory. For simplicity we will denote the observable corresponding to $k = 0$ by $\mathcal{O}(x)$. The most general observable which we will consider

in this paper will have the form:

$$\mathcal{O}(x_1) \cdots \mathcal{O}(x_r) I(\Sigma_1) \cdots I(\Sigma_s). \tag{2.20}$$

Correlation functions involving operators of the form (2.20) vanish unless the following selection rule holds:

$$4r + 2s = \dim \mathcal{M}_{\mathrm{NA}}. \tag{2.21}$$

where $\dim \mathcal{M}_{\mathrm{NA}}$ is given in (2.5) with $N = 2$. These correlation functions of the topological field theory are interpreted mathematically as intersection forms in the moduli space. The operator $\mathcal{O}$ represents a cohomology class of degree four, and $I(\Sigma)$ represents a cohomology class of degree two. The condition (2.21) simply says that the integral of these differential forms vanishes unless the total degree equals the dimension of the moduli space. This has a natural interpretation in field-theoretical terms [12]. The dimension of the moduli space corresponds to the index of the operator:

$$T = ds \oplus C^\dagger : \ \Omega^1(X, \mathbf{g}_E) \oplus \Gamma(X, S^+ \otimes E) \longrightarrow \Omega^0(X, \mathbf{g}_E) \oplus \Omega^{2,+}(X, \mathbf{g}_E) \oplus \Gamma(X, S^- \otimes E), \tag{2.22}$$

which gives the instanton deformation complex for the moduli problem of nonabelian monopoles. But this is also the operator associated to the grassmannian fields in (2.16), and its index gives the anomaly in the ghost number. The selection rule (2.21) is therefore the 't Hooft rule which says that fermionic zero modes in the path integral measure should be soaked up in the correlation functions.

As it is usual in quantum field theory, we will group all the correlation functions of operators like (2.20) in a generating function,

$$\langle \exp(\sum_a \alpha_a I(\Sigma_a) + \mu \mathcal{O}) \rangle, \tag{2.23}$$

summed over instanton numbers of the bundle E. In (2.23) the Σ_a denote a basis of the two-dimensional homology of X, and therefore $a = 1, \cdots, \dim\ H_2(X, \mathbf{Z})$.

The action (2.16) can be obtained from the twist of $N = 2$ supersymmetric Yang-Mills theory with gauge group $SU(N)$ coupled to an $N = 2$ hypermultiplet in the fundamental representation [17,18,19,20,22]. The basic idea involved in the twisting is the following. In $\mathbf{R}^4$ the global symmetry group of $N = 2$ supersymmetry is $\mathcal{H} = SU(2)_L \otimes SU(2)_R \otimes SU(2)_I \otimes U(1)_{\mathcal{R}}$, where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group and $SU(2)_I$ and $U(1)_{\mathcal{R}}$ are internal symmetry groups. The supercharges Q_α^i and $\overline{Q}_{\dot\alpha i}$ of $N = 2$ supersymmetry transform under $\mathcal{H}$ as $(1/2, 0, 1/2)^1$ and $(0, 1/2, 1/2)^{-1}$ respectively. The twist consists in considering as the rotation group the group $\mathcal{K}' = SU(2)'_L \otimes SU(2)_R$ where $SU(2)'_L$ is the diagonal subgroup of $SU(2)_L \otimes SU(2)_I$. Under the new global symmetry group $\mathcal{H}' = \mathcal{K}' \otimes U(1)_{\mathcal{R}}$ the supercharges transforms as $(1/2, 1/2)^{-1} \oplus (1, 0)^1 \oplus (0, 0)^1$. In the proccess of twisting the isospin index i becomes a spinor index, $Q_\alpha^i \to Q_\alpha{}^\beta$ and $Q_{\dot\alpha i} \to Q_{\dot\alpha\beta}$, and the trace of $Q_\alpha{}^\beta$, $Q = Q_\alpha{}^\alpha$, becomes a $(0, 0)$ rotation invariant operator.

If there is no $N = 2$ central extension, from the supersymmetry algebra follows that Q obeys $Q^2 = 0$. This operator can be regarded as a BRST operator and the $U(1)_{\mathcal{R}}$ charges as ghost numbers. In $\mathbf{R}^4$ the original and the twisted theory are the same. For other manifolds the two theories are certainly different since the stress tensor changes. On the other hand, due to the fact that the operator Q is an scalar, it also exists for arbitrary four manifolds. The existence of this operator is what gives the topological character to twisted theories.

We will begin briefly describing the twist of $N = 2$ supersymmetric Yang-Mills theory. Then, we will consider the case of its coupling to an $N = 2$ hypermultiplet. The field content of the minimal $N = 2$ supersymmetric Yang-Mills theory with gauge group G is the following: a gauge field $A_{\alpha\dot\alpha}$ (using the notation in [21], $A_{\alpha\dot\alpha} = e_\mu^m(\sigma_m)_{\alpha\dot\alpha}$), fermions λ_α^i and $\overline{\lambda}_{\dot\alpha i}$, a complex scalar B, and an auxiliary field D_{ij} (symmetric in i and j). All these fields are considered in the adjoint

representation of the gauge group G. Under the twisting these fields become:

$$
\begin{aligned}
A_{\alpha\dot\alpha} \;\; (1/2,1/2,0)^0 &\longrightarrow A_{\alpha\dot\alpha} \;\; (1/2,1/2)^0, \\
\lambda^i_\alpha \;\; (1/2,0,1/2)^{-1} &\longrightarrow \eta \;\; (0,0)^{-1}, \;\; \chi_{\alpha\beta} \;\; (1,0)^{-1}, \\
\overline{\lambda}_{\dot\alpha i} \;\; (0,1/2,1/2)^1 &\longrightarrow \psi_{\alpha\dot\alpha} \;\; (1/2,1/2)^1, \\
B \;\; (0,0,0)^{-2} &\longrightarrow \lambda \;\; (0,0)^{-2}, \\
B^\dagger \;\; (0,0,0) &\longrightarrow \phi \;\; (0,0)^2, \\
D_{ij} \;\; (0,0,1)^0 &\longrightarrow H_{\alpha\beta} \;\; (1,0)^0,
\end{aligned}
\tag{2.24}
$$

where we have indicated the quantum numbers carried out by the fields relative to the group $\mathcal{H}$ before the twisting, and to the group $\mathcal{H}'$ after the twisitng. Notice that the fields $\chi_{\alpha\beta}$ and $H_{\alpha\beta}$ are symmetric in α and β and therefore they can be regarded as components of two self-dual two-forms. The definitions of the twisted fields in terms of the untwisted ones are the obvious ones from (2.24). The only ones which need clarification are the conventions taken for η and χ_{12}. Our choice is:

$$
\lambda^1_1 = \frac{1}{2}\eta - \chi_{12}, \qquad \lambda^2_2 = \frac{1}{2}\eta + \chi_{12}.
\tag{2.25}
$$

The Q-transformations of the twisted fields can be obtained very simply from the $N = 2$ supersymmetry transformations. These last transformations are generated by the operator $\eta^\alpha_i Q^i_\alpha + \overline{\eta}^{\dot\alpha i}\overline{Q}_{\dot\alpha i}$ where η^α_i and $\overline{\eta}^{\dot\alpha i}$ are anticommuting parameters. To get the Q-transformations of the fields one must consider $\overline{\eta}^{\dot\alpha i} = 0$ and replace $\eta^\alpha_i \to \rho\delta^\alpha_\beta$, being ρ an arbitrary scalar anticommuting parameter. The twisting leads to the transformations (2.11) for the twisted fields on the right hand side of (2.24), and to a twisted action which turns out to be the action S_0 in (2.13) for some value of a plus a term of the form (2.15).

Before discussing the coupling of an $N = 2$ hypermultiplet let us make a few comments on the twisting from a $N = 1$ superspace point of view. In $N = 1$ superspace only one of the supersymmetries is manifest, and therefore the $N = 1$ superfields do not have well defined quantum numbers respect to the internal

$SU(2)_I$ symmetry. The $N = 2$ supersymmetric multiplet contains an $N = 1$ vector multiplet and an $N = 1$ chiral multiplet. These multiplets are described in $N = 1$ superspace in terms of $N = 1$ superfields W_α and Φ satisfying the constraints $\overline{D}_{\dot\alpha} W_\alpha = 0$, $D^\alpha W_\alpha + \overline{D}^{\dot\alpha} \overline{W}_{\dot\alpha} = 0$ and $\overline{D}_{\dot\alpha}\Phi = 0$, where D_α and $\overline{D}_{\dot\alpha}$ are $N = 1$ superspace covariant derivatives (we use the conventions in [30]). The $N = 1$ superfields W_α and Φ have $U(1)_{\mathcal{R}}$ charges -1 and -2 respectively. The component fields of the $N = 1$ superfields W_α and Φ are:

$$
\begin{aligned}
W_\alpha, \ \overline{W}_{\dot\alpha} &\longrightarrow \ A_{\alpha\dot\alpha}, \ \lambda_\alpha^1, \ \overline{\lambda}_{\dot\alpha 1}, \ D_{12}, \\
\Phi, \ \Phi^\dagger &\longrightarrow \ B, \ \lambda_\alpha^2, \ D_{11}, \ B^\dagger, \ \overline{\lambda}_{\dot\alpha 2}, \ D_{22}.
\end{aligned}
\tag{2.26}
$$

The $U(1)_{\mathcal{R}}$ transformations of the $N = 1$ superfields are:

$$
W_\alpha \to e^{-i\phi} W_\alpha(e^{i\phi}\theta), \quad \text{and} \quad \Phi \to e^{-2i\phi}\Phi(e^{i\phi}\theta).
\tag{2.27}
$$

Notice that these transformations are consistent with the assignment in (2.24).

In $N = 1$ superspace the action of $N = 2$ supersymmetric Yang-Mills theory takes the form:

$$
\int d^4x \, d^2\theta d^2\overline{\theta} \, \Phi^\dagger e^V \Phi + \int d^4x \, d^2\theta \, \text{Tr}(W^\alpha W_\alpha) + \int d^4x \, d^2\overline{\theta} \, \text{Tr}(\overline{W}^{\dot\alpha}\overline{W}_{\dot\alpha}),
\tag{2.28}
$$

where V is the vector superpotential. An important feature of this action is that due to the constraint $D^\alpha W_\alpha + \overline{D}^{\dot\alpha}\overline{W}_{\dot\alpha} = 0$ the last two terms in (2.28) differ by a term which is proportional to the second Chern class.

$N = 2$ matter is usually represented by $N = 2$ hypermultiplets. The hypermultiplet contains a complex scalar isodoublet q^i, fermions $\psi_{q\alpha}$, $\psi_{\tilde{q}\alpha}$, $\overline{\psi}_{q\dot\alpha}$, $\overline{\psi}_{\tilde{q}\dot\alpha}$, and a complex scalar isodoublet auxiliary field F^i. The fields q^i, $\psi_{q\alpha}$, $\overline{\psi}_{\tilde{q}\dot\alpha}$, and F^i are in the fundamental representation of the gauge group, while the fields $q_i^\dagger$, $\psi_{\tilde{q}\alpha}$, $\overline{\psi}_{q\dot\alpha}$, and $F_i^\dagger$ are in the conjugate representation. Under the twisting these fields

become:

$$\begin{aligned}
q^i \ (0,0,1/2)^0 &\longrightarrow M^\alpha \ (1/2,0)^0, \\
\psi_{q\alpha} \ (1/2,0,0)^1 &\longrightarrow \mu_\alpha \ (1/2,0)^1, \\
\overline{\psi}_{\tilde{q}\dot{\alpha}} \ (0,1/2,0)^{-1} &\longrightarrow v_{\dot{\alpha}} \ (0,1/2)^{-1}, \\
F^i \ (0,0,1/2)^2 &\longrightarrow K^\alpha \ (1/2,0)^2, \\
q_i^\dagger \ (0,0,1/2)^0 &\longrightarrow \overline{M}_\alpha \ (1/2,0)^0, \\
\overline{\psi}_{q\dot{\alpha}} \ (0,1/2,0)^{-1} &\longrightarrow \overline{v}_{\dot{\alpha}} \ (0,1/2)^{-1}, \\
\psi_{\tilde{q}\alpha} \ (1/2,0,0)^1 &\longrightarrow \overline{\mu}_\alpha \ (1/2,0)^1, \\
F_i^\dagger \ (0,0,1/2)^{-2} &\longrightarrow \overline{K}_\alpha \ (1/2,0)^{-2}.
\end{aligned} \tag{2.29}$$

The Q transformations of the twisted fields are obtained in the same way as in the case of the $N = 2$ vector multiplet. The resulting transformations, however, are not the ones in (2.11). First of all notice that the auxiliary fields of the twisted theory in (2.29) are different than the auxiliary fields in (2.11). This is a first hint on the existence of some differences between the theory in (2.12) and the twisted theory. Auxiliary fields are useful in supersymmetry because they permit to close the supersymmey off-shell. In this section we have considered a version of the $N = 2$ hypermultiplet which contains a minimal set of auxiliary fields. For this version, however, there is a non-trivial central charge Z. This is an inconvenient for the twisting because then one finds $Q^2 = Z$ instead of $Q^2 = 0$. On the other hand, if one disregards this problem and goes along considering the twisted theory, it turns out that after integrating the auxiliary fields the resulting action of the twisted theory is just the action S in (2.12) for a specific value of a:

$$a = \frac{1}{\sqrt{2}}. \tag{2.30}$$

This can be obtained comparing (2.12) to the action originated from the twisting of the $N = 2$ theory presented in [19]. This equivalence proves that in the twisted theory the auxiliary content of the theory and the Q-transformations involving these fields can be changed in such a way that an off-shell action can be written

as a Q-exact quantity and, furthermore, $Q^2 = 0$. In other words, one can indeed affirm that the twisted theory is topological. This was observed for the first time in [17].

Let us briefly describe the $N = 2$ hypermultiplet from the point of view of $N = 1$ superspace. This multiplet contains two $N = 1$ chiral multiplets and therefore it can be described by two $N = 1$ chiral superfields Q (this Q should not be confused with the BRST operator) and $\widetilde{Q}$, i.e., these superfields satisfy the constraints $\overline{D}_{\dot\alpha} Q = 0$ and $\overline{D}_{\dot\alpha} \widetilde{Q} = 0$. They have $U(1)_{\mathcal{R}}$ charge 0. While the superfield Q is in the fundamental representation of the gauge group, the superfield $\widetilde{Q}$ is in the corresponding conjugate representation. The component fields of these $N = 1$ superfields are:

$$
\begin{aligned}
Q, \ Q^\dagger \ &\longrightarrow \ q^1, \ \psi_{q\alpha} \ F^1, \ q_1^\dagger \ \overline{\psi}_{\tilde q \dot\alpha}, \ F_1^\dagger, \\
\widetilde{Q}, \ \widetilde{Q}^\dagger \ &\longrightarrow \ q_2^\dagger, \ \psi_{\tilde q \alpha}, \ F_2^\dagger, \ q^2, \ \overline{\psi}_{q\dot\alpha}, \ F^2.
\end{aligned}
\tag{2.31}
$$

Again, notice that the $U(1)_{\mathcal{R}}$ transformations of the $N = 1$ superfields,

$$
Q \to Q(\mathrm{e}^{i\phi}), \quad \text{and} \quad \widetilde{Q} \to \widetilde{Q}(\mathrm{e}^{i\phi}),
\tag{2.32}
$$

are consistent with the assignment in (2.29).

In $N = 1$ superspace the action for the $N = 2$ hypermultiplet coupled to $N = 2$ supersymmetric Yang-Mills takes the form:

$$
\int d^4x \, d^2\theta \, d^2\overline{\theta} \, (Q^\dagger \mathrm{e}^V Q + \widetilde{Q}^\dagger \mathrm{e}^{-V} \widetilde{Q}) + \sqrt{2} \int d^4x \, d^2\theta \, \widetilde{Q} \Phi Q + \sqrt{2} \int d^4x \, d^2\overline{\theta} \, \widetilde{Q}^\dagger \Phi^\dagger Q^\dagger.
\tag{2.33}
$$

Notice that the last two terms are consistent with the fact that while Φ is in the adjoint representation of the gauge group, the superfields Q and $\widetilde{Q}$ are in the fundamental and in its conjugate, respectively.

2.4. TWIST ON KÄHLER MANIFOLDS

In this subsection we describe some aspects of the theory under consideration when the manifold X is Kähler. Work on Donaldson-Witten theory on Kähler manifolds can be found in [14,31]. When the metric on the four-manifold X is Kähler the global group $SU(2)_L \otimes SU(2)_R$ becomes $U(1)_L \otimes SU(2)_R$ being $U(1)_L$ a subgroup of $SU(2)_L$. The two dimensional representation of $SU(2)_L$ decomposes under $U(1)_L$ as a sum of one dimensional representations. This means that the components M^1 and M^2 transform in definite representations of $U(1)_L$ with opposite charges. In other words, $S^+ \otimes E$ has a decomposition into $(K^{\frac{1}{2}} \otimes E) \oplus (K^{-\frac{1}{2}} \otimes E)$, where K is the canonical bundle. The complex structure on X allows to have well defined complex forms of type (p, q). We define this complex structure stating the following assignment:

$$
\begin{aligned}
(\sigma_m)_{1\dot{\alpha}}\, dx^m, &\quad \text{type } (0,1), \\
(\sigma_m)_{2\dot{\alpha}}\, dx^m, &\quad \text{type } (1,0).
\end{aligned}
\tag{2.34}
$$

This implies that $(\sigma_{mn})_{\alpha\beta}\, dx^m \wedge dx^n$ can be regarded as a $(0,2)$ form when $\alpha = \beta = 1$, as $(2,0)$ form when $\alpha = \beta = 2$, and as a $(1,1)$ form when $\alpha = 1$, $\beta = 2$.

Let us recall that in the process of twisting the BRST operator Q was obtained form the supersymmetric charge Q^i_α after identifying $Q^i_\alpha \longrightarrow Q_\alpha{}^\beta$ and then performing the sum $Q = Q_1{}^1 + Q_2{}^2$. In the Kähler case, each of the components, $Q_1{}^1$ and $Q_2{}^2$, transforms under definite $U(1)_L$ representations and therefore one can define two BRST charges $Q_1 = Q_1{}^1$ and $Q_2 = Q_2{}^2$. Of course, from the supersymmetry algebra follows that $Q_1^2 = 0$ and $Q_2^2 = 0$. Furthermore, from their construction: $Q = Q_1 + Q_2$. The action of each of these two operators on the fields is easily obtained from the supersymmetry transformations. One just have to set $\eta^{\dot{\alpha}i} = 0$ and, for Q_1 $\eta_1^\alpha = \rho_1 \delta_1^\alpha$ and $\eta_2^\alpha = 0$, while, for Q_2, $\eta_1^\alpha = 0$ and $\eta_2^\alpha = \rho_2 \delta_2^\alpha$. From the point of view of $N = 2$ superspace the operators Q_1 and Q_2 can be regarded as a specific derivative respect to some of the θ's. In the formulation of the theory on $N = 1$ superspace the operator Q_1 can be identified as the derivative

respect to θ_1. This observation will be very helpfull in proving the invariance under Q_1 of the twisted theories.

On a Kähler manifold each of the fields on the right hand side of (2.24) splits into fields which can be thought as components of forms of type (p, q). For the matter fields on the right hand side of (2.29) one just has the standard decomposition of $S^+ \otimes E$ into $(K^{\frac{1}{2}} \otimes E) \oplus (K^{-\frac{1}{2}} \otimes E)$. For example for the field M^α one has:

$$M^\alpha \to M^1 \in \Gamma(K^{\frac{1}{2}} \otimes E), \quad M^2 \in \Gamma(K^{-\frac{1}{2}} \otimes E),$$
$$\overline{M}_\alpha \to \overline{M}_1 \in \Gamma(K^{-\frac{1}{2}} \otimes \overline{E}), \quad \overline{M}_2 \in \Gamma(K^{\frac{1}{2}} \otimes \overline{E}).$$
(2.35)

A similar decomposition holds for the rest of the fields in $\Gamma(S^+ \otimes E)$ on the right hand side of (2.29). Notice that the product of an element of $\Gamma(K^{\frac{1}{2}} \otimes E)$ times an element of $\Gamma(K^{\frac{1}{2}} \otimes \overline{E})$ is a gauge invariant form of type $(2, 0)$. From the identifications in (2.29) and (2.31) follows that the first component of $\widetilde{Q}Q$, i.e., $\widetilde{Q}Q| = q_2^\dagger q^1 = \overline{M_2}^\dagger M^1$ is a $(2, 0)$ form. Therefore, superpotentials of the form $\widetilde{Q}Q$, or $\widetilde{Q}\Phi Q$ as the one in (2.33) can be regarded as $(2, 0)$-forms. This is consistent with the observation made in [14, 16] that superpotential terms of a twisted theory on a Kähler manifold must transform as $(2, 0)$-forms.

Since the twisted theory obtained fron (2.28) and (2.33) and the topological theory (2.12) are equivalent on-shell for $a = 1/\sqrt{2}$ we will work out the on-shell Q_1-transformations for this case. Notice that only if $a = 1/\sqrt{2}$ in (2.12) one can guarantee Q_1-invariance. The Q_1-transformations for the twisted fields in the $N = 2$ vector multiplet turn out to be:

$$[Q_1, A_{1\dot\alpha}] = \psi_{1\dot\alpha}, \qquad [Q_1, \lambda] = \frac{1}{2}\eta + \chi_{12},$$
$$[Q_1, A_{2\dot\alpha}] = 0, \qquad \{Q_1, \frac{1}{2}\eta + \chi_{12}\} = 0,$$
$$\{Q_1, \psi_{1\dot\alpha}\} = 0, \qquad \{Q_1, \frac{1}{2}\eta^a - \chi_{12}^a\} = +2i(F_{12}^{a+} + \frac{i}{2}(\overline{M}_1 T^a M_2 + \overline{M}_2 T^a M_1)) - \frac{i}{2}[\lambda, \phi]^a,$$
$$\{Q_1, \psi_{2\dot\alpha}\} = D_{2\dot\alpha}\phi, \qquad \{Q_1, \chi_{11}^a\} = \overline{M}_1 T^a M_1,$$
$$[Q_1, \phi] = 0, \qquad \{Q_1, \chi_{22}\} = -iF_{22}^+,$$
(2.36)

19

where we have used that the generators of the gauge group are normalized in such a way that $\text{Tr}(T^a T^b) = \delta^{ab}$. For the matter fields one finds:

$$
\begin{aligned}
&[Q_1, M^1] = \mu^1, && [Q_1, \overline{M}_1] = 0, \\
&[Q_1, M^2] = 0, && [Q_1, \overline{M}_2] = \overline{\mu}_2, \\
&\{Q_1, \mu^1\} = 0, && \{Q_1, \overline{\mu}_1\} = \phi \overline{M}_1, && (2.37) \\
&\{Q_1, \mu^2\} = \phi M^2, && \{Q_1, \overline{\mu}_2\} = 0, \\
&\{Q_1, v_{\dot\alpha}\} = D_{2\dot\alpha} M^2, && \{Q_1, \overline{v}_{\dot\alpha}\} = \overline{D}_{2\dot\alpha} \overline{M}_1.
\end{aligned}
$$

It is straightforward to verify that indeed $Q_1^2 = 0$ on-shell after working out the transformations of the different components of $F_{\alpha\beta}^+$:

$$
[Q_1, F_{11}^+] = i D_{1\dot\sigma} \psi_1{}^{\dot\sigma}, \quad [Q_1, F_{12}^+] = \frac{i}{2} D_{2\dot\sigma} \psi_1{}^{\dot\sigma}, \quad [Q_1, F_{22}^+] = 0. \tag{2.38}
$$

The Q_2-transformations are easily computed from (2.36), (2.37) and (2.11) after using $Q = Q_1 + Q_2$. The action S in (2.12) for $a = 1/\sqrt{2}$ is invariant under both, Q_1 and Q_2 symmetries. This can be verified explicitly or just using the following argument based on $N = 1$ superspace. On the one hand, the topological action (2.12) can be regarded as a twisted version of the sum of the $N = 1$ superspace actions (2.28) and (2.33). On the other hand, the Q_1 operator is equivalent to a θ_1-derivative. Acting with this derivative on (2.28) and (2.33) one gets zero: for the terms involving chiral fields one ends with two many θ-derivatives, while for the other terms one just gets a total derivative after using the fact that $[D_\alpha, \overline{D}^2] = i\partial_{\alpha\dot\alpha} \overline{D}^{\dot\alpha}$.

It is often convenient to regard the the observables $I(\Sigma)$ in (2.19) in terms of the Poincaré dual of the homology cycle Σ:

$$
I(\Sigma) = \int_\Sigma \mathcal{O}^{(k)} = \int_X \mathcal{O}^{(2)} \wedge [\Sigma], \tag{2.39}
$$

where $[\Sigma]$ denotes the Poincaré dual. On Kähler manifolds, $I(\Sigma)$ can be decomposed in three different types of operators depending on which holomorphic part of

$\mathcal{O}^{(2)}$ is taken into account. If only the (p, q) part $(p + q = 2)$ of $\mathcal{O}^{(2)}$ is considered we will denote the corresponding operator by $I^{p,q}(\Sigma)$. For example, for the $(1,1)$ part:

$$I^{1,1}(\Sigma) = \frac{1}{2} \int_X e(i\phi F_{12} - \frac{1}{2}\psi_{1\dot\alpha}\psi_2^{\dot\alpha})[\Sigma]_{12} = \frac{1}{2} \int_X e(iB^\dagger F_{12} - \frac{1}{2}\overline{\lambda}_{1\dot\alpha}\overline{\lambda}_2^{\dot\alpha})[\Sigma]_{12}, \quad (2.40)$$

where in the last step we have used (2.24), and we have denoted by Σ_{12} the $(1,1)$ part of Σ.

2.5. The perturbed massive theory on Kähler manifolds

One of the main ingredients in the analysis made by Witten in [14] is the existence of a perturbation of the twisted $N = 2$ Yang-Mills theory on Kähler manifolds which while preserving the topological character of the twisted theory it allows to regard the theory from an untwisted point of view as an $N = 1$ supersymmetric theory. Witten achieved this demostrating that on a Kähler manifold it is possible to add an $N = 1$ supersymmetric mass-like term for the chiral superfield Φ while keeping the topological character of the theory. In this subsection we will show that this is also possible for the topological quantum field theory which describes non-abelian monopoles. Notice that, as we need the superpotentials to transform as $(2,0)$-forms, to generate a mass term for Φ we must pick a holomorphic $(2,0)$-form on X. This is not always possible on an arbitrary Kähler manifold, but as we are assuming $b_2^+ > 1$ in order to have a well-defined moduli problem, we guarantee that $H^{2,0}(X) \neq 0$ and hence that such a form exists.

Let us consider a holomorphic $(2, 0)$ form ω on X. Its only non-vanishing component is:

$$\omega_{11} = (\sigma_{lk})_{11}\omega_{mn}\epsilon^{lkmn}. \quad (2.41)$$

We will denote the unique non-vanishing component of the $(0, 2)$ form $\overline{\omega}$, conjugate to ω, by $\overline{\omega}_{22}$ ($\overline{\omega}_{22} = (\omega_{11})^*$)

Following [14] we begin making a perturbation of the action S in (2.16) by adding a term of the form,

$$I(\omega) = \int_X \mathcal{O}^{(2)} \wedge \omega, \tag{2.42}$$

where $I(\omega)$ is the observable defined in (2.19). In (2.42) we are denoting the Poincaré dual to the holomorphic $(2,0)$ form ω by the symbol ω as well. Using the Q_1-transformations (2.36) and (2.37), this term can be written as:

$$I(\omega) = -\frac{1}{2} \int_X d^4x \, e \, \omega_{11} \mathrm{Tr}(\frac{1}{2}\psi_{2\dot\alpha}\psi_2^{\dot\alpha}) + \{Q_1, -\frac{1}{2} \int d^4x \, e \, \omega_{11} \mathrm{Tr}(\phi\chi_{22})\}. \tag{2.43}$$

The first part of this term indicates some progress towards the construction of an $N = 1$ mass-like term. However, (2.43) is not invariant under Q_1. Contrary to the case of the theory without matter fields this term is not even Q_1-invariant on-shell. One can remedy this problem if instead of introducing $I(\omega)$ one considers:

$$\tilde{I}(\omega) = I(\omega) - \frac{1}{2} \int_X d^4x \, e \, \omega_{11} \overline{M}_2 \phi M_2. \tag{2.44}$$

Indeed, $\{Q_1, \tilde{I}(\omega)\}$ turns out to be proportional to the field equation resulting after making a variation respect to χ^{22} in the twisted action S in (2.12) (with $a = 1/\sqrt{2}$) .

The term $\tilde{I}(\omega)$ implies further progress towards the perturbation by an $N = 1$ supersymmetric mass term. Notice that from an $N = 1$ superspace point of view we intend to obtain a term of the form,

$$m \int d^4x \, d^2\theta \mathrm{Tr}(\Phi^2) + \overline{m} \int d^4x \, d^2\overline{\theta} \mathrm{Tr}(\Phi^{\dagger^2}). \tag{2.45}$$

This type of term, added to a theory which already has the last two terms in (2.33), leads, when written in component fields, to terms like the one added to $I(\omega)$ in (2.44).

To make further progress in the perturbation towards an $N = 1$ supersymmetric mass term while maintaining the Q_1 symmetry we will modify the Q_1-transformation of χ_{11} in the following way:

$$\{Q_1, \chi_{11}^a\} \longrightarrow \{Q_1', \chi_{11}^a\} = \overline{M}_1 T^a M_1 + \omega_{11}\phi^a, \tag{2.46}$$

while for the rest of the fields the action of Q_1 and Q_1' remains the same. Notice that still one has $Q_1'^2 = 0$ on-shell.

Under Q_1' the action S is not invariant. However, one can verify that now the perturbed action $S + \tilde{I}(\omega)$ is invariant and not a field equation as before. On the other hand, adding a Q_1'-exact term will keep the Q_1'-invariance of the theory. It is rather remarkable that adding just the term,

$$-\frac{1}{2}\Big\{Q_1', \int d^4x\, e\, \overline{\omega}_{22} \mathrm{Tr}(\lambda\chi_{11})\Big\}, \tag{2.47}$$

one finds that the perturbed action is just the action S plus an $N = 1$ supersymmetric mass term for the chiral superfield Φ:

$$
\begin{aligned}
S + \tilde{I}&(\omega) + \{Q_1', ...\} \\
&= S - \frac{1}{2}\int_X d^4x\, e\Big(\overline{\omega}_{22}\mathrm{Tr}\big((\frac{1}{2}\eta + \chi_{12})\chi_{11}\big) + \omega_{11}\mathrm{Tr}(\frac{1}{2}\psi_{2\dot\alpha}\psi_2{}^{\dot\alpha})\Big) + \int_X d^4x\, e\,\omega_{11}\overline{\omega}_{22}\mathrm{Tr}(\lambda\phi) \\
&\quad - \frac{1}{2}\int_X d^4x\, e(\omega_{11}\overline{M}_2\phi M_2 + \overline{\omega}_{22}\overline{M}_1\lambda M_1).
\end{aligned}
\tag{2.48}
$$

This perturbation of the action S contains all the terms present in the $N = 1$ supersymmetric mass term (2.45) after setting $m = \omega_{11}$ and integrating out the auxiliary fields. Writing the twisted fields in terms of the untwisted ones the form of (2.45) in component fields is obtained. Recall that according to (2.24) and (2.25), $\lambda_2^2 = \frac{1}{2}\eta + \chi_{12}$, $\lambda_1^2 = \chi_{11}$, $\lambda_{2\dot\alpha} = \psi_{2\dot\alpha}$, $B^\dagger = \phi$, and $B = \lambda$. For the matter fields one can read their untwisted counterparts from (2.29).

23

Our analysis implies that if one denotes correlation functions of observables in the twisted theory by $\langle A_1 \cdots A_n \rangle$, and in the perturbed theory by $\langle A_1 \cdots A_n \rangle_1$, the relation between them is:

$$\langle A_1 \cdots A_n \rangle_1 = \langle A_1 \cdots A_n e^{-\tilde{I}(\omega)} \rangle. \tag{2.49}$$

As argued in [14], given some homology cycles Σ, it can be assumed that near their intersection they look like holomorphically embedded Riemann surfaces. This means that actually the only relevant part of the two-form operators entering (2.49) are of type $(1, 1)$. Precisely those are the two-form operators invariant under Q'_1. This follows trivially using (2.36) in (2.40). As the zero-form observables in (2.49) are also invariant under Q'_1 one can regard the right hand side of (2.49) as a topological quantum field theory whose BRST operator is Q'_1 and its action is $S + \tilde{I}(\omega)$.

The effect of an extra term $I(\omega)$ in the action of Donaldson-Witten theory was studied by Witten in [14]. He showed that its effect on correlators of observables can be described as a shift on the parameters corresponding to the observables containing two-form operators. We will finish this section showing that the relevant contribution from $\tilde{I}(\omega)$ in (2.49) and from $I(\omega)$ in the case of Donaldson-Witten theory is the same. Therefore, in our theory the effect of the presence of $\tilde{I}(\omega)$ in (2.49) is also a shift in those parameters.

The quantity $\tilde{I}(\omega)$ can be written as,

$$
\begin{aligned}
\tilde{I}(\omega) =& \frac{1}{2} \int_X d^4x \, e \, \omega_{11} \left(i\phi^a (F_{22}^{a+} + i\overline{M}_2 T^a M_2) - \text{Tr}(\frac{1}{2}\psi_{2\dot\alpha}\psi_2{}^{\dot\alpha}) \right) \\
=& -\frac{1}{2} \int_X d^4x \, e \, \omega_{11} \left(\text{Tr}(\{Q, \phi\chi_{22}\} + \frac{1}{2}\psi_{2\dot\alpha}\psi_2{}^{\dot\alpha}) \right),
\end{aligned}
\tag{2.50}
$$

after using (2.11). This means that the vacuum expectation values on the right

24

hand side of (2.49) can be written as

$$\langle A_1 \cdots A_n e^{J(\omega)} \rangle. \tag{2.51}$$

where,

$$J(\omega) = \frac{1}{4} \int_X d^4x \, e \, \omega_{11} \mathrm{Tr}(\psi_{2\dot\alpha}\psi_2{}^{\dot\alpha}). \tag{2.52}$$

This is precisely the same expression that one obtains in the case of Donaldson-Witten theory. Notice that in that case F_{22}^+ is Q-exact and one has the same Q-transformations as in our theory for the field $\psi_{a\dot\alpha}$.

Another argument to show that the presence of the term involving the massive fields in $\tilde{I}(\omega)$ is irrelevant is just to point out that the contributions from the functional integral on the right hand side of (2.49) are localized on configurations satisfying the monopole equations. As shown in [21] the $(0, 2)$ part of those equations implies $\overline{M}_2 T M_2 = 0$.

3. Vacuum structure of the $N = 2$ and $N = 1$ theories

As we have seen, when we perturb the original $N = 2$ theory to a $N = 1$ theory on a Kähler manifold, the resulting theory preserves the topological symmetry and one can compute the topological correlation functions in the $N = 1$ theory. If this theory has a mass gap and presents topological invariance, the only relevant information we need in this computation is the structure of the $N = 1$ vacua and their symmetries, as was shown by Witten in [14]. When we consider $N = 2$ pure Yang-Mills theory perturbed by a mass term for the chiral multiplet Φ, we know that at low energies we are reduced to a $N = 1$ Yang-Mills theory. This theory is supposed to have a mass gap and its vacua have the symmetry pattern coming from spontaneous chiral symmetry breaking [15]. But the standard conjectures about the structure of vacua of this theory can be obtained from the structure of the quantum moduli space of vacua of the corresponding $N = 2$ theory, as it has been shown in [6]. The same method can be applied to the $N = 2$ theory with matter perturbed by the $N = 1$ mass term for Φ [7]. In this section we will use the information about symmetries and vacua of the $N = 2$ theory [7] to show that in fact the $N = 1$ theory we are dealing with has a mass gap and we will obtain a precise description of its vacua.

$N = 2$ supersymmetric QCD with gauge group $SU(N_c)$ and N_f hypermultiplets in the fundamental representation of the gauge group has the $U(1)_{\mathcal{R}}$ symmetry described in (2.27) and (2.32):

$$\begin{aligned}
W_\alpha &\longrightarrow e^{-i\phi}W_\alpha(e^{i\phi}\theta), & Q^i &\longrightarrow Q^i(e^{i\phi}\theta), \\
\Phi &\longrightarrow e^{-2i\phi}\Phi(e^{i\phi}\theta), & \widetilde{Q}_{\widetilde{i}} &\longrightarrow \widetilde{Q}_{\widetilde{i}}(e^{i\phi}\theta).
\end{aligned} \tag{3.1}$$

In component fields the corresponding transformations can be read from (2.24) and (2.29):

$$\begin{aligned}
\lambda^1,\ \lambda^2 &\longrightarrow e^{-i\phi}\lambda^1,\ e^{-i\phi}\lambda^2, \\
B &\longrightarrow e^{-2i\phi}B, \\
\psi_{qi},\ \psi_{\widetilde{q}\widetilde{i}} &\longrightarrow e^{i\phi}\psi_{qi},\ e^{i\phi}\psi_{\widetilde{q}\widetilde{i}}.
\end{aligned} \tag{3.2}$$

This symmetry is anomalous because of instanton effects. The anomaly is $4N_c - 2N_f$ ($2N_c$ from λ^1 and λ^2, which live in the adjoint representation of the gauge group, and 2 from each couple of fermions ψ_q, $\psi_{\tilde{q}}$ in the hypermultiplet). In the case we are dealing with, namely $N_c = 2$ and $N_f = 1$ (which gives the $SU(2)$ monopole equations) the anomaly is 6 and we should expect the $\mathbf{Z}_6$ anomaly-free discrete subgroup:

$$
\begin{aligned}
\lambda^1, \ \lambda^2 &\longrightarrow \mathrm{e}^{-\frac{i\pi}{3}}\lambda^1, \ \mathrm{e}^{-\frac{i\pi}{3}}\lambda^2, \\
B &\longrightarrow \mathrm{e}^{-\frac{2i\pi}{3}}B, \\
\psi_q, \ \psi_{\tilde{q}} &\longrightarrow \mathrm{e}^{\frac{i\pi}{3}}\psi_q, \ \mathrm{e}^{\frac{i\pi}{3}}\psi_{\tilde{q}}.
\end{aligned}
\tag{3.3}
$$

However, since we are considering one hypermultiplet in the fundamental representation of $SU(2)$, we must take into account that the quark Q and the antiquark $\widetilde{Q}$ live in isomorphic representations of the gauge group (for $SU(2)$, $\mathbf{2} \simeq \bar{\mathbf{2}}$). As a consequence of this isomorphism, and when the matter fields are massless, we have a parity symmetry ρ interchanging the quark and the antiquark:

$$
\rho : Q \leftrightarrow \widetilde{Q}.
\tag{3.4}
$$

This symmetry is anomalous, as can be seen from the 't Hooft interaction term,

$$
(\lambda^1)^4(\lambda^2)^4\psi_q\psi_{\tilde{q}}.
\tag{3.5}
$$

Nevertheless, one can combine the ρ symmetry with the square root of $\mathbf{Z}_6$ in (3.3) to obtain an anomaly-free $\mathbf{Z}_{12}$ subgroup.

Under the $\mathbf{Z}_{12}$ symmetry the quantity $u = \mathrm{Tr}B^2$ transforms as $u \to \mathrm{e}^{-2\pi i/3}u$ and gives a global $\mathbf{Z}_3$ symmetry on the u plane. This plane parametrizes in fact the moduli space of vacua of the theory. Classically, $SU(2)$ is broken to $U(1)$ for $u \neq 0$, and at $u = 0$ the gauge symmetry is unbroken as the gluons become massless. Quantum mechanically, the picture that emerges is very different: there are three singularities interchanged by the $\mathbf{Z}_3$ symmetry. These singularities are points where magnetic monopoles or dyons become massless, and the description

based on a low-energy effective action which includes only the photon multiplet of the unbroken $U(1)$ breaks down: an additional massless hypermultiplet must be included near each singularity [7]. The effective theory becomes therefore $N = 2$ supersymmetric QED with a massless hypermultiplet.

The superpotential of the resulting $N = 2$ supersymmetric QED in terms of $N = 1$ superfields has the following form:

$$W_{\mathrm{M}} = \sqrt{2}AM\widetilde{M}. \tag{3.6}$$

In this expression A denotes the $N = 1$ chiral multiplet of the $N = 2$ Yang-Mills field (it is the abelian analogue of Φ). The superfields M and $\widetilde{M}$ are the $N = 1$ chiral multiplets which represent the $N = 2$ hypermultiplet in the abelian case. They have opposite charges. We are interested in the vacuum structure of the $N = 1$ theory which is obtained when one adds a mass term $m\mathrm{Tr}\Phi^2$ to the $N = 2$ theory with matter. For this one can use the effective low-energy action, as it is shown in [6, 7]. The effective contribution of the mass term for Φ in the low-energy theory can be represented by an additional term in the superpotential $W_{\mathrm{eff}} = mU$, where U is a chiral superfield whose first component is the operator u. The vacua of the $N = 1$ theory are given by the critical points of the superpotential, up to complexified $U(1)$ gauge transformations (this is equivalent to set the D terms to zero and divide by $U(1)$). At non-singular points of the moduli space, $W = W_{\mathrm{eff}}$ and therefore, if one supposes that $du \neq 0$, there are no supersymmetric ground states at all. The only points in the quantum moduli space of vacua of the $N = 2$ theory which give rise to $N = 1$ vacua are precisely the singularities where monopoles become massless. In this case $W = W_{\mathrm{M}} + W_{\mathrm{eff}}$ and there are critical points where magnetic monopoles get an expectation value. Hence, the resulting $N = 1$ theory has three vacua related by the $\mathbf{Z}_3$ symmetry of the u-plane. In these points one can also check that there is mass gap and condensation of monopoles.

Because of the mass gap of the $N = 1$ theory we can use the physical properties of this kind of theories to evaluate the correlation functions. We also know that

this theory has three vacua, but to have a clear picture of their symmetries we need the resulting $U(1)_\mathcal{R}$ symmetry of the perturbed theory. Notice that the mass term for the Φ field breaks the second transformation in (3.1) due to the presence of the fermionic fields λ^2, as can be seen from (2.48). Thus under the new $U(1)_\mathcal{R}$ symmetry we must have:

$$\Phi \longrightarrow \mathrm{e}^{-i\phi}\Phi(\mathrm{e}^{i\phi}\theta), \tag{3.7}$$

and this in turn imposes, because of the superpotential term, the following transformation for the matter fields:

$$
\begin{aligned}
Q &\longrightarrow \mathrm{e}^{-i\phi/2}Q(\mathrm{e}^{i\phi}\theta), \\
\widetilde{Q} &\longrightarrow \mathrm{e}^{-i\phi/2}\widetilde{Q}(\mathrm{e}^{i\phi}\theta).
\end{aligned}
\tag{3.8}
$$

Rescaling the charges to make them integers, we have the following $U(1)_\mathcal{R}$ symmetry for the perturbed theory in terms of components fields:

$$
\begin{aligned}
\lambda^1,\ B &\longrightarrow \mathrm{e}^{-2i\phi}\lambda^1,\ \mathrm{e}^{-2i\phi}B, \\
q,\ \tilde{q} &\longrightarrow \mathrm{e}^{-i\phi}q,\ \mathrm{e}^{-i\phi}\tilde{q}, \\
\psi_q,\ \psi_{\tilde{q}} &\longrightarrow \mathrm{e}^{i\phi}\psi_q,\ \mathrm{e}^{i\phi}\psi_{\tilde{q}}.
\end{aligned}
\tag{3.9}
$$

The anomaly-free discrete subgroup of the transformations (3.9) is $\mathbf{Z}_6$. However, one must take into account the ρ symmetry (3.4), as the addition of the mass term for Φ doesn't break it. Again, we have an enhancement of the discrete symmetry to $\mathbf{Z}_{12}$. The resulting transformations are:

$$
\begin{aligned}
\lambda^1,\ B &\longrightarrow \mathrm{e}^{-\pi i/3}\lambda^1,\ \mathrm{e}^{-\pi i/3}B, \\
q &\longrightarrow \mathrm{e}^{-\pi i/6}\tilde{q}, \quad \tilde{q} \longrightarrow \mathrm{e}^{-\pi i/6}q, \\
\psi_q &\longrightarrow \mathrm{e}^{\pi i/6}\psi_{\tilde{q}}, \quad \psi_{\tilde{q}} \longrightarrow \mathrm{e}^{\pi i/6}\psi_q.
\end{aligned}
\tag{3.10}
$$

These transformations leave invariant the 't Hooft term $(\lambda^1)^4\psi_q\psi_{\tilde{q}}$. We know that this theory has only three vacua, and therefore there must be spontaneous

symmetry breaking of (3.10), as it happens in the pure $N = 1$ Yang-Mills theory. To identify the pattern of this breaking, notice that these vacua are labeled by the order parameter $u = \text{Tr}B^2$. It is easy to see that the unbroken symmetry which gives this vacuum structure is:

$$\lambda^1 \longrightarrow -\lambda^1,$$
$$\psi_q \longrightarrow i\psi_{\tilde{q}}, \qquad (3.11)$$
$$\psi_{\tilde{q}} \longrightarrow i\psi_q.$$

This is precisely the maximal subgroup of (3.10) which allows fermion masses for λ^1 and for ψ_q, $\psi_{\tilde{q}}$ (notice that the mass term for the matter fields changes its sign under the parity symmetry involved in (3.11)). The spontaneous chiral symmetry breaking in (3.11) is induced by a vacuum expectation value of the gauge invariant order parameter $X = \widetilde{Q}Q$ as in [32].

4. Computation of the polynomial invariants

In this section we compute the topological invariants corresponding to $SU(2)$ monopoles by two different methods. The first one is based on the abstract approach developed in [14] and is valid only for Kähler manifods. The second method, which is valid for arbitrary spin manifolds, uses electric-magnetic duality [6,7] and is inspired by the approach developed in [8].

4.1. KÄHLER MANIFOLDS

Now that we have the information about the vacuum structure of the $N = 1$ theory and their symmetries we can compute the correlation functions of the topological theory on a Kähler, spin manifold. Because of the presence of the mass gap most of the arguments of [14] go through. Although the structure of the mass perturbation on a Kähler manifold introduces some subtleties which we will consider later (the cosmic string theory), in a first approach the correlation functions take the form:

$$\langle\exp(\sum_a \alpha_a I(\Sigma_a) + \mu\mathcal{O})\rangle = \sum_\rho C_\rho\exp(\gamma_\rho v^2 + \mu\langle\rho|\mathcal{O}|\rho\rangle). \tag{4.1}$$

In this expression the sum is over the three vacua $|\rho\rangle$ labeled by the index $\rho = 1, 2, 3$; $v = \sum_a \alpha_a[\Sigma_a]$, where $[\Sigma_a]$ is the cohomology class Poincaré dual to Σ_a, and $v^2 = \sum_{a,b} \alpha_a\alpha_b\sharp(\Sigma_a \cap \Sigma_b)$, where $\sharp(\Sigma_a \cap \Sigma_b)$ is the intersection number of Σ_a and Σ_b. The constant C_ρ is the partition function in the ρ vacuum, and the mass gap and topological character of the theory imply that it must have the structure:

$$C_\rho = \exp(a_\rho\chi + b_\rho\sigma). \tag{4.2}$$

The constants which appear in (4.1) are not independent because the theory has a $\mathbf{Z}_3$ broken symmetry which relates the three vacua and is given by (3.10). First let us work out the relation between the C_ρ. As these constants are given by

the partition function of the theory at different vacua, and the vacua are related by a non-anomalous symmetry, one should think that they are equal. But actually, as we are working now on a curved four-manifold, the anomalies have gravitational contributions which were not taken into account in section 3 (where $X = \mathbf{R}^4$), and the path integral measure does change. We must take into account also the new geometrical content of the fields after twisting. The field $\overline{\lambda}_1$ is now a $(1,0)$-form, and λ^1 contains a $(2,0)$-form part, a $(1,1)$ part and a scalar part. The operator relating them is:

$$\partial \oplus p_+ \overline{\partial} \oplus \partial^\dagger : \ \Omega^{1,0} \longrightarrow \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^0, \tag{4.3}$$

and its index is given by half the dimension of $\mathcal{M}_{\mathrm{ASD}}$ in (2.6) (notice that the complex conjugate of this operator gives the one for $\overline{\lambda}_2$, which is a $(0,1)$-form, so both indices are equal and the sum of them gives the index of the original ASD complex (2.7)). Therefore the anomaly due to the first transformation in (3.10) is:

$$e^{\frac{\pi i}{3}\{4k - \frac{3}{4}(\chi + \sigma)\}}. \tag{4.4}$$

We must take into account also the transformation of the matter fermions. After twisting they are spinors, and we have the correspondence $\psi_q \to \mu$, $\psi_{\tilde{q}} \to \bar{\mu}$ (see (2.29)). Notice that, due to the ρ symmetry in (3.4), there is an additional contribution to the anomaly coming from this transformation, as we saw in (3.5). Now we must also compute the gravitational part and obtain the total anomaly (a similar problem is addressed in section 4.4 of [33]). The path integral measure for the twisted matter fermions can be written as:

$$\prod_I d\mu_I d\bar{\mu}_I \prod_J dv_J d\bar{v}_J, \tag{4.5}$$

where the index $I = 1, \cdots, \nu_+$ refers to the μ^α zero modes (of positive chirality) and the index $J = 1, \cdots, \nu_-$ to the $v_{\dot{\alpha}}$ zero modes (of negative chirality). Under

the transformation in (3.10), the measure (4.5) transforms as:

$$(-1)^{\nu_+ + \nu_-} e^{-\frac{\pi i}{6} 2(k + \frac{\sigma}{4})} = (-1)^{-k - \frac{\sigma}{4}} e^{-\frac{\pi i}{3}(k + \frac{\sigma}{4})}, \tag{4.6}$$

where we have taken into account that $\nu_+ - \nu_- =$ index $D = -k - \sigma/4$, according to (2.8). Putting together both factors we obtain:

$$(-1)^\Delta e^{-\frac{\pi i}{12}\sigma}, \tag{4.7}$$

where Δ was introduced in (2.10) and we have used that $\sigma \equiv 0$ mod 8. Notice that the k dependence has dropped out, because the symmetry in (3.10) is not anomalous under Yang-Mills instantons, and (4.7) contains only the gravitational contribution to the anomaly. The result, in terms of the constants C_ρ, is:

$$C_2 = (-1)^\Delta e^{-\frac{\pi i}{12}\sigma} C_1, \quad C_3 = e^{-\frac{\pi i}{6}\sigma} C_1. \tag{4.8}$$

We would also like to relate the constants γ_ρ and the expectation values $\langle \rho | \mathcal{O} | \rho \rangle$ in (4.1) for the different vacua. This is easily done taken into account the transformations of the corresponding observables under the symmetry (3.10). As it is argued in [14], for the observables $I(\Sigma)$ on a Kähler manifold one can consider only the $(1,1)$ part. If we call α the generator of the discrete symmetry in (3.10) in the operator formalism, after taking into account (2.40), one finds the following relations:

$$\alpha |\rho\rangle = |\rho + 1\rangle, \quad \rho \equiv 1 \mod 3,$$
$$\alpha \mathcal{O} \alpha^{-1} = e^{\frac{2\pi i}{3}} \mathcal{O}, \quad \alpha I^{1,1}(\Sigma) \alpha^{-1} = e^{\frac{\pi i}{3}} I^{1,1}(\Sigma), \tag{4.9}$$

which lead to:

$$\gamma_2 = e^{-\frac{2\pi i}{3}} \gamma_1, \quad \gamma_3 = e^{-\frac{4\pi i}{3}} \gamma_1,$$
$$\langle 2 | \mathcal{O} | 2 \rangle = e^{-\frac{2\pi i}{3}} \langle 1 | \mathcal{O} | 1 \rangle, \quad \langle 3 | \mathcal{O} | 3 \rangle = e^{-\frac{4\pi i}{3}} \langle 1 | \mathcal{O} | 1 \rangle. \tag{4.10}$$

With these relations we have determined completely the bulk structure of the vacua, which comes from the underlying $N = 1$ theory.

One has to take into account however that the mass perturbation which gives this theory was done with a $(2,0)$ holomorphic form ω, and the mass will vanish when this form does. In general, ω vanishes on a divisor C representing the canonical class of X. The simplest case is the one in which C is a union of disjoint Riemann surfaces C_y of multiplicity $r_y = 1$ (*i.e.*, ω has simple zeroes along this components), and therefore the canonical divisor of X can be written as,

$$c_1(K) = \sum_y [C_y]. \tag{4.11}$$

As discussed in [14, 16], near these surfaces C_y we have an effective two-dimensional theory (the cosmic string theory) with additional symmetry breaking. In particular, along the worldsheets of the strings C_y each bulk vacuum bifurcates according to a new pattern of symmetry breaking. As in [14, 16], we will assume that each bulk vacuum gives two vacua along the string, and therefore that there is spontaneous symmetry breaking of $\lambda^1 \to -\lambda^1$. As we will see, this assumption is the most natural one from several points of view. First of all, the contributions from the new vacua cooperate with the bulk structure in such a way that the resulting expression has the adequate properties. Second, with this assumption, the final expression can be naturally understood as a consequence of electric-magnetic duality of the underlying $N = 2$ theory. Finally, as we will describe in sect. 5, where we consider the $N = 2$ supersymmetric theory with a massive hypermultiplet, one can show that, although the bulk structure of vacua of that theory is different from the one under consideration, the "internal" structure of each vacuum corresponds in fact to a two-fold bifurcation.

Let us briefly explain, following [14], what is the effect of the new vacua in the computation of the correlation functions. Each bulk vacuum $|\rho\rangle$ leads to two vacua of the cosmic string theory $|\rho+\rangle$, $|\rho-\rangle$, which are related by the broken symmetry α^3. The surfaces C_y give new contributions to the correlators through their intersection with the surfaces Σ_a. The observables $I(\Sigma_a)$ will be described by $\sharp(\Sigma_a \cap C_y)V_y$, where V_y is the insertion of a cosmic string operator V on C_y which

has the same quantum numbers of $I^{1,1}(\Sigma_a)$. From (2.40) and (3.10) follows that it transforms under α^3 as:

$$\alpha^3 V \alpha^{-3} = -V. \tag{4.12}$$

Now, for a given bulk vacuum $|\rho\rangle$ we must take into account its bifurcation along the diferent surfaces C_y, and compute the vacuum expectation values of $\exp(\sum_a \alpha_a I(\Sigma_a))$. The result is [14]:

$$\prod_y \Big(\exp(\phi_y\langle\rho+|V|\rho+\rangle) + t_y\exp(-\phi_y\langle\rho+|V|\rho+\rangle)\Big). \tag{4.13}$$

In this expression, $\phi_y = \sum_a \alpha_a \sharp(\Sigma_a \cap C_y)$ and the factor t_y is similar to (4.7) and comes from an anomaly in the two-dimensional effective theory. It is given by:

$$t_y = (-1)^{\epsilon_y}, \tag{4.14}$$

where ϵ_y is 0 (1) if the spin bundle of C_y is even (odd). The ϵ_y verify [14]:

$$\Delta + \sum_y \epsilon_y \equiv 0 \mod 2. \tag{4.15}$$

At this point we have all the information that we need to compute the polynomial invariants for $SU(2)$ monopoles on Kähler, spin four-manifolds. Notice that the result will involve unknown constants which should be fixed by comparing to mathematical computations of these invariants. These constants are universal, in the sense that they depend only on the dynamics of the physical theory (as shown in [14]) and not on the particular manifold we are considering. If we denote $C = C_1$, $\gamma = \gamma_1$, $\langle 1|\mathcal{O}|1\rangle = \langle\mathcal{O}\rangle$ and $\langle 1+|V|1+\rangle = \langle V\rangle$, the expression for the

polynomial invariants reads:

$$
C\Bigg(\exp(\gamma v^2 + \mu\langle\mathcal{O}\rangle)\prod_y\Big(\exp(\langle V\rangle\phi_y) + t_y\exp(-\langle V\rangle\phi_y)\Big)
$$
$$
+ (-1)^\Delta e^{-\frac{\pi i}{12}\sigma}\exp\Big(e^{-\frac{2\pi i}{3}}(\gamma v^2 + \mu\langle\mathcal{O}\rangle)\Big)\prod_y\Big(\exp(e^{-\frac{\pi i}{3}}\langle V\rangle\phi_y) + t_y\exp(-e^{-\frac{\pi i}{3}}\langle V\rangle\phi_y\Big)
$$
$$
+ e^{-\frac{\pi i}{6}\sigma}\exp\Big(e^{-\frac{4\pi i}{3}}(\gamma v^2 + \mu\langle\mathcal{O}\rangle)\Big)\prod_y\Big(\exp(e^{-\frac{2\pi i}{3}}\langle V\rangle\phi_y) + t_y\exp(-e^{-\frac{2\pi i}{3}}\langle V\rangle\phi_y)\Big)\Bigg).
$$
$$(4.16)$$

In order to check some of the properties of (4.16) we will express it in a more convenient way. Notice that because of (4.14) and (4.15) we can extract the factor t_y in the second summand of (4.16) and cancel the factor $(-1)^\Delta$. Using some straightforward algebra and the fact that $\sigma \equiv 0 \mod 8$, (4.16) can be rewritten as:

$$
\langle\exp(\sum_a \alpha_a I(\Sigma_a) + \mu\mathcal{O})\rangle
$$
$$
= C\Bigg(\exp(\gamma v^2 + \mu\langle\mathcal{O}\rangle)\prod_y\Big(\exp(\langle V\rangle\phi_y) + t_y\exp(-\langle V\rangle\phi_y)\Big)
$$
$$
+ e^{-\frac{\pi i}{6}\sigma}\exp\Big(-e^{-\frac{\pi i}{3}}(\gamma v^2 + \mu\langle\mathcal{O}\rangle)\Big)\prod_y\Big(\exp(e^{-\frac{2\pi i}{3}}\langle V\rangle\phi_y) + t_y\exp(-e^{-\frac{2\pi i}{3}}\langle V\rangle\phi_y)\Big)
$$
$$
+ e^{-\frac{\pi i}{3}\sigma}\exp\Big(-e^{\frac{\pi i}{3}}(\gamma v^2 + \mu\langle\mathcal{O}\rangle)\Big)\prod_y\Big(\exp(e^{-\frac{4\pi i}{3}}\langle V\rangle\phi_y) + t_y\exp(-e^{-\frac{4\pi i}{3}}\langle V\rangle\phi_y)\Big)\Bigg)
$$
$$(4.17)$$

where the second summand of (4.16) is now the last one. This is our final expression for the polynomial invariants associated to $SU(2)$ monopoles on Kähler, spin manifolds whose canonical divisor can be written as in (4.11). The result is obviously real, as the first summand in (4.17) is real and the second one is the complex conjugate of the third one.

Another check of (4.17) is the following. As we noticed in sect. 2, a product of r observables $\mathcal{O}$ and s observables $I(\Sigma)$ has ghost number $4r + 2s$, and this must

36

equal the dimension of the moduli space for some instanton number k. In terms of Δ we have the selection rule (2.21):

$$4r + 2s = \dim \, \mathcal{M}_{\mathrm{NA}} = 6(k - \Delta) - \frac{\sigma}{2}, \tag{4.18}$$

i.e., if we suppose that the α_a are of degree 2 and μ of degree 4, in the expansion of (4.17) we can only find terms whose degree is congruent to $-\sigma/2$ mod 6. This is easily checked. If we consider the terms of fixed degree $4r + 2s$ we see that they can be grouped in terms with the same coefficient, given by:

$$1 + e^{-\frac{\pi i}{6}\sigma} e^{-\frac{\pi i}{3}(4r+2s)} + e^{-\frac{\pi i}{3}\sigma} e^{-2\frac{\pi i}{3}(4r+2s)}. \tag{4.19}$$

This is a geometrical series whose sum is zero unless $e^{-\frac{\pi i}{6}\sigma - \frac{\pi i}{3}(4r+2s)} = 1$, which gives precisely the condition we were looking for. Notice that to obtain the well-behaved expression (4.17) from (4.16) the key point is that the contributions from the cosmic string theory have the form (4.13). This is what allows to drop out the factor $(-1)^{\Delta}$ which comes from the bulk structure and suggests that the pattern of bifurcation of vacua along the cosmic string is the right one.

Another point of interest is that, according to our expression (4.17), the generating function for the correlation functions $f = \langle \exp(\sum_a \alpha_a I(\Sigma_a) + \mu \mathcal{O}) \rangle$ verifies the equation:

$$\frac{\partial^3 f}{\partial \mu^3} = \langle \mathcal{O} \rangle^3 f, \tag{4.20}$$

which seems to be the adequate generalization to our moduli problem of the simple type condition which appears in Donaldson theory [34]. Physically, the order of this equation is clearly related to the number of singularities which appear in the quantum moduli space of vacua. It would be interesting to have a clear picture of the mathematical meaning of this generalized simple type condition as well as to know what is the form it takes in other moduli problems.

4.2. General spin manifolds

In the previous section we have computed the polynomial invariants for $SU(2)$ monopoles on Kähler, spin manifolds. The fact that we have a Kähler structure allows one, as we have seen in sect. 2, to perform the computation in the $N = 1$ theory. In this section we will show that one can use electric-magnetic duality and the $U(1)_\mathcal{R}$ symmetry of the original $N = 2$ theory to obtain expressions which are valid on a general spin manifold X, as it happens in Donaldson theory [8]. As it has been shown in [6, 7], at every point in the quantum moduli space of vacua of the $N = 2$ Yang-Mills theory (the u-plane) there is a low-energy abelian $N = 2$ effective theory which can also be twisted to give a topological field theory. At a generic point the only light degree of freedom is the $U(1)$ gauge field which survives after gauge symmetry breaking, and the twisting of this theory would give (as it should be clear from sect. 2) the moduli problem of abelian instantons on X. At the singularities new massless states (monopoles or dyons) appear which must be included in the low-energy lagrangian. The resulting effective theory is $N = 2$ QED with a certain number of massless hypermultiplets. For the pure $N = 2$ Yang-Mills theory and the theory with $N_f = 1$ which we have been considering, there is a single hypermultiplet at every singularity [7]. In these cases, the twisted theory near these points gives the abelian monopole equations of [8]. Such a theory has been constructed in [13]. In principle, when computing a correlation function of the original, "microscopic" twisted theory, one should integrate over the u-plane. However, the moduli problem in Donaldson theory and in the non-abelian monopole theory (as it has been argued in [21]) are well defined only for manifolds with $b_2^+ > 1$. This condition means that there are no abelian instantons on X for a generic metric, and therefore one expect contributions only from the singularities, as the moduli space of the twisted effective, "macroscopic" theory is empty for the other points in the u-plane. This is consistent with the $N = 1$ point of view. Once we know the contribution from one of the singularities, the other contributions can be obtained through the underlying microscopic $U(1)_\mathcal{R}$ symmetry of the $N = 2$ theory.

Let us implement this picture in our problem. The quantum moduli space of vacua has three singularities, as we have recalled in sect. 2. Each of them corresponds to a single state becoming massless. The charges of the three different states are $(n_m, n_e) = (1, 0)$, $(1, 1)$, $(1, 2)$, where n_m denotes the magnetic charge and n_e the electric one [7]. Consider the singularity associated to the magnetic monopole, with charge $(1, 0)$. The low-energy effective theory after twisting gives the moduli problem of abelian monopoles, and as the observables of the theory are those of the pure Yang-Mills case, we should expect that the contribution from this singularity to a correlation function is the same which appears in the $N = 2$ theory without matter coming from the singularity at $u = \Lambda_0^2$. Recall from [8] that the abelian monopole theory is defined in terms of a complex line bundle L (in the spin case) or equivalently by a class $x = -2c_1(L)$. When the moduli space associated to the abelian monopole equation has zero dimension, x satisfies:

$$x^2 = 2\chi + 3\sigma, \tag{4.21}$$

and the partition function of this theory is denoted by n_x. The x such that $n_x \neq 0$ are called basic classes, and they must verify the condition (4.21). According to [8], the contribution from the vacua at $u = \Lambda_0^2$ is given by:

$$\exp(\gamma v^2 + \mu \langle \mathcal{O} \rangle) \sum_x n_x e^{\langle V \rangle v \cdot x}, \tag{4.22}$$

where $v \cdot x = \sum_a \alpha_a \sharp(\Sigma_a \cap x)$ and we have included the new universal constants which also appear in (4.17). The sum is over all the basic classes. Now, to get the contributions from the other two vacua we can use the $U(1)_\mathcal{R}$ symmetry given in (3.1) and (3.2). The resulting $\mathbf{Z}_6$ anomaly-free discrete subgroup in (3.3). Notice that if we use (3.3) the transformation of the order parameter u is $u \rightarrow e^{-\frac{4\pi i}{3}} u$. This is still a $\mathbf{Z}_3$ symmetry of the u-plane which goes through all the singularities, and we won't need to implement the additional symmetry (3.4).

At this point the computation becomes very similar to the one we did for the $N = 1$ theory. First we must take into account the gravitational contribution of

the anomaly and the geometrical character of the fields after twisting. The fields λ^1, λ^2, $\overline{\lambda}_1$ and $\overline{\lambda}_2$ give now the whole ASD complex (2.7) and the anomaly is the square of (4.4). For the matter fermions the anomaly is given by $\frac{2\pi i}{3}$index D. The total contribution is:

$$e^{\frac{\pi i}{3}\{8k-\frac{3}{2}(\chi+\sigma)\}-\frac{\pi i}{3}(2k+\frac{\sigma}{2})} = e^{-\frac{\pi i \sigma}{6}}. \tag{4.23}$$

This is the anomaly we obtained for the third $N = 1$ vacuum, for it is the square of (4.7). We see that, as it happens in the pure $N = 2$ Yang-Mills theory [14], the $U(1)_\mathcal{R}$ symmetries of the $N = 1$ and the $N = 2$ theory, which are certainly different, work in such a way that after twisting one obtains the same contribution for the gravitational part of the anomaly.

To implement the symmetry under consideration in the observables we need the action of the generator of (3.3), call it again α, on them. One obtains:

$$\alpha \mathcal{O} \alpha^{-1} = -e^{\frac{\pi i}{3}} \mathcal{O}, \quad \alpha I(\Sigma) \alpha^{-1} = e^{\frac{2\pi i}{3}} I(\Sigma). \tag{4.24}$$

Now we can apply these transformations to (4.22), as we did in the Kähler case, to obtain:

$$
\begin{aligned}
&\langle \exp(\sum_a \alpha_a I(\Sigma_a) + \mu \mathcal{O}) \rangle \\
&= C\Bigg(\exp(\gamma v^2 + \mu\langle\mathcal{O}\rangle) \sum_x n_x \exp(\langle V\rangle v \cdot x) \\
&\quad + e^{-\frac{\pi i}{6}\sigma} \exp\Big(-e^{-\frac{\pi i}{3}}(\gamma v^2 + \mu\langle\mathcal{O}\rangle) \Big) \sum_x n_x \exp(e^{-\frac{2\pi i}{3}}\langle V\rangle v \cdot x) \\
&\quad + e^{-\frac{\pi i}{3}\sigma} \exp\Big(-e^{\frac{\pi i}{3}}(\gamma v^2 + \mu\langle\mathcal{O}\rangle) \Big) \sum_x n_x \exp(e^{-\frac{4\pi i}{3}}\langle V\rangle v \cdot x) \Bigg).
\end{aligned}
\tag{4.25}
$$

This is our final expression for the polynomial invariants associated to $SU(2)$ monopoles on a spin manifold X with $b_2^+ > 1$.

Using the abelian monopole equations it is easy to show that, when X is Kähler and its canonical divisor is of the form (4.11) with disjoint C_y, one recovers (4.17) from (4.25). In such a situation the basic classes are given by [8]:

$$x_{(\rho_1,\cdots,\rho_n)} = \sum_y \rho_y [C_y],$$ (4.26)

and each $\rho_y = \pm 1$. The corresponding n_x are:

$$n_x = \prod_y t_y^{s_y},$$ (4.27)

where $s_y = (1 - \rho_y)/2$ and the t_y are given in (4.14). A simple computation leads to,

$$\sum_x n_x \exp(\langle V \rangle v \cdot x) = \sum_{\rho_y} \prod_y t_y^{s_y} \exp(\sum_y \langle V \rangle \rho_y \phi_y) = \prod_y \left(\exp(\langle V \rangle \phi_y) + t_y \exp(-\langle V \rangle \phi_y) \right),$$ (4.28)

which shows that in this case (4.25) becomes (4.17).

Our last comment concerns the rôle of (4.20) in the above computation. As it is argued in [8] for Donaldson theory, the simple type condition guarantees that the operator $\mathrm{Tr}(B^\dagger)^2$ can be replaced by c-numbers in the evaluation of the correlation functions, and there are no contributions coming from other operators of the effective theory. The equivalent condition in our case is certainly (4.20), and therefore the expression (4.25) should be valid for spin manifolds with $b_2^+ > 1$ whose polynomial invariants verify this constraint.

5. The massive theory

In this section we will make some observations concerning $N = 2$ QCD with one massive hypermultiplet. The superpotential (3.6) now becomes:

$$W = \sqrt{2}\widetilde{Q}\Phi Q + m\widetilde{Q}Q. \tag{5.1}$$

This theory can be twisted as in section 2.3, but the resulting theory is not topological because of the mass term, and can be understood in fact as a deformation of the theory we have been studying. Notice that the $U(1)_\mathcal{R}$ symmetry of the non-massive theory is completely broken by the presence of the mass term, and therefore one would expect that the quantum moduli space of this theory has a singularity structure very different from the original one. These singularities can be analyzed along the lines proposed in [7].

Classically the B field gets an expectation value characterized by a complex parameter a, and one can write $B = a\sigma_3$ with $\sigma_3 = \mathrm{diag}(1, -1)$. If we write $Q = (Q^1, Q^2)$, $\widetilde{Q} = (\widetilde{Q}^1, \widetilde{Q}^2)$ and expand around this vacuum, it is easy to see from (5.1) that the first component of the hypermultiplet gets a mass $m + \sqrt{2}a$, while the second one gets a mass $m - \sqrt{2}a$. One finds a classical singularity at $a = -m/\sqrt{2}$ where the first component of the hypermultiplet become massless: as we have a new light degree of freedom, the description based on the pure $N = 2$ abelian theory breaks down. If we consider that $m \gg \Lambda_1$, where Λ_1 is the dynamically generated mass of the theory, the singularity is in the semiclassical region (because $u \approx 2a^2 = m^2 \gg \Lambda_1$) and it persists in the quantum theory. For $u \ll m^2$ all the quarks are massive and can be integrated out. The low-energy theory is the pure $N = 2$ Yang-Mills, which has two singularities at $u = \pm\Lambda_0^2$ [6], where Λ_0 is the scale of the low-energy theory and is related to Λ_1 by $\Lambda_0^4 = m\Lambda_1^3$ [7]. The conclusion of this analysis is that the massive theory has three singularities, as the non-massive one, and obviously there is not a discrete symmetry coming from an $U(1)_\mathcal{R}$ relating them.

If we perturb the massive theory with a mass term for the Φ superfield, as we have done in the computation of the polynomial invariants for the non-massive theory, we obtain an $N = 1$ theory with three vacua coming from the singularities in the quantum moduli space of the underlying $N = 2$ theory. Here we will obtain these three vacua for the twisted massive theory on a Kähler, spin manifold X. Instead of considering the effective theory, as in sect. 3, we will follow the procedure used in [16] to obtain similar issues in $N = 4$ Yang-Mills theory, and we will analyze the $N = 1$ superpotential which is obtained after perturbation. Notice that, as the mass term for the matter hypermultiplet is Q_1'-closed (following the same kind of arguments used in sect. 2), we have the same situation with respect to the topological symmetry that the one we had in the non-massive theory. The only term breaking the topological invariance of the resulting $N = 1$ theory will be again the mass term.

Recall that, as we are on a Kähler manifold, the terms in the superpotential must be $(2, 0)$ forms. The mass perturbation for the Φ superfield must be done with a global ω, which is a holomorphic section of the canonical bundle K over X. In this paper we take it satisfying (4.11). The superpotential reads:

$$W = \widetilde{Q}\Phi Q + m\widetilde{Q}Q + \omega \mathrm{Tr}\Phi^2. \tag{5.2}$$

In order to obtain the critical points of this function we have to solve the following equations:

$$
\begin{aligned}
\frac{\partial W}{\partial Q_a} &= \sqrt{2}\widetilde{Q}_b\Phi_{ab} + m\widetilde{Q}_a = 0, \\
\frac{\partial W}{\partial \widetilde{Q}_a} &= \sqrt{2}\Phi_{ab}Q_a + mQ_a = 0, \\
\frac{\partial W}{\partial \Phi_{ab}} &= \sqrt{2}\widetilde{Q}_aQ_b + 2\overline{\omega}\Phi_{ab} = 0, \quad a \neq b, \\
\frac{\partial W}{\partial \Phi_{11}} &= \sqrt{2}(\widetilde{Q}_1Q_1 - \widetilde{Q}_2Q_2) + 4\omega\Phi_{11} = 0.
\end{aligned}
\tag{5.3}
$$

An obvious solution of the equations (5.3) is the trivial one, with $\Phi = Q = \widetilde{Q} = 0$. This corresponds to the trivial embedding solution in [16], and gives at low energies

43

the pure $N = 1$ Yang-Mills theory, with two different vacua. This is the same structure we found in the quantum moduli space of vacua of the $N = 2$ massive theory for the low-energy behaviour. Because of the third singularity, we should expect a non-trivial critical point for (5.2). The first equation of (5.3) tells us that Q is an eigenvector of Φ with eigenvalue $-\sqrt{2}m/2$. As Φ is a traceles matrix, their eigenvalues must be $-\sqrt{2}m/2$ and $\sqrt{2}m/2$. Now recall that we must quotient the solutions of (5.3) by the group of complexified gauge transformations, *i.e.* transformations in $Sl(2, \mathbf{C})$. We can use this gauge freedom to write Φ in diagonal form:

$$\Phi \longrightarrow M\Phi M^{-1} = -\sqrt{2}m/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M \in Sl(2, \mathbf{C}). \tag{5.4}$$

This means that at this critical point Φ breaks the gauge group down to $U(1)$, as it should be expected from the description by means of the $N = 2$ theory.

In the twisted theory on a Kähler manifold we have $Q = (\alpha_1, \alpha_2) \in K^{1/2} \otimes E$ and similarly $\widetilde{Q} = (\beta_1, \beta_2) \in K^{1/2} \otimes \overline{E}$ (here we denote by capital letters also the first component of the chiral superfields Q, $\widetilde{Q}$ and Φ). The vacuum of the theory must have zero action, and this requires Φ to be holomorphic because of its kinetic energy term. But if a holomorphic section of the adjoint gauge bundle ad$E$ splits everywhere as in (5.4), then $E = L \oplus L^{-1}$, with L a holomorphic line bundle. Let us analyze the remaining equations in (5.3). After conjugation by M, Q and $\widetilde{Q}$ verify:

$$\begin{aligned} \widetilde{Q}_1 Q_2 &= \widetilde{Q}_2 Q_1 = 0, \\ \widetilde{Q}_1 Q_1 - \widetilde{Q}_2 Q_2 &= 2m\omega. \end{aligned} \tag{5.5}$$

Choosing $Q_1 \neq 0$, we have $\widetilde{Q}_2 = Q_2 = 0$, and $\widetilde{Q}_1 Q_1 = 2m\omega$. Because of the splitting of E, we have $\alpha \equiv \alpha_1 \in K^{1/2} \otimes L$, and $\beta \equiv \beta_1 \in K^{1/2} \otimes L^{-1}$. The last equation in (5.5) gives:

$$\alpha\beta = 2m\omega, \tag{5.6}$$

which is essentially the (perturbed) abelian monopole equation of [8] on a Kähler

manifold (notice also that in order to obtain a vacuum α and β must be holomorphic, as required in [8]).

The above result confirms the picture for the low-energy theories associated to the singularities of the $N = 2$ massive theory: at these points the light degrees of freedom are a matter hypermultiplet and the $N = 2$ photon, and the effective theory should be $N = 2$ QED with one matter field. This theory, after twisting, gives the moduli problem of abelian monopoles, and its vacua correspond to the solutions of these equations. This is precisely what we have obtained as the critical points of W. Notice that the two bulk vacua associated to the trivial solution, and corresponding to pure $N = 1$ Yang-Mills, have the same internal vacuum strucure given by the solution to the monopole equations. This is because, when the canonical divisor has the form (4.11), the solutions of (5.6) correspond precisely to the two-fold bifurcation of each bulk vacuum along the cosmic strings C_y, as it is shown in [8].

One should expect that the "internal" vacuum structure associated to the bulk vacua of the massive theory are equivalent to the ones arising in the non-massive theory. This is clear from the $N = 2$ point of view, where the low-energy effective theories are equivalent, at least in the limit of very large Λ. Therefore, the analysis that we have done supports the assumption we made in sect. 4 about the cosmic string theory. Notice that we have not used duality arguments in this analysis, but rather we have checked their predictions.

6. Conclusions

In this paper we have computed the polynomial invariants associated to the moduli space of $SU(2)$ monopoles on four-dimensional spin manifolds, with the monopole fields in the fundamental representation of the gauge group. Our computation is based on the exact results about the quantum moduli space of vacua of the corresponding $N = 2$ and $N = 1$ supersymmetric theories, and follows the lines of [14] and [8]. The resulting expressions (4.25) and (4.17) can be written in terms of Seiberg-Witten invariants, and therefore the first conclusion of our analysis is that these invariants underlie not only Donaldson theory, but also the generalization of this theory presented in [21]. This is a striking result, as the moduli space of $SU(2)$ monopoles seems at first view very different from the moduli space associated to the abelian monopole equations. Certainly it should be very interesting to have a mathematical understanding of this fact, as well as expressions for the monopole invariants computed by mathematical methods in order to compare them to our results.

The picture which emerges from our computation is that non-perturbative methods in supersymmetric gauge theories are not only an extremely powerfool tool to obtain topological invariants, but also to relate very different moduli problems in four-dimensional geometry. Notice that the information about the quantum moduli space of vacua in [6,7] is obtained integrating out the massive excitations of the original field theory, in order to obtain low-energy effective descriptions. The topological information of the twisted, microscopic theories seems to be encoded in only two parameters of the non-perturbative results: the number of singularities in the moduli space of vacua (related by an anomaly-free discrete subgroup) and the number of hypermultiplets becoming massless at these singularities. It seems that different four-dimensional moduli problems can be in the same "universality class" when considered from the point of view of the underlying supersymmetric theories. Therefore, using non-perturbative results in the physical theories, one should be able to identify truly basic topological invariants characterizing a whole family of

moduli problems. According to our results, the $SU(2)$ Donaldson invariants and the $SU(2)$ monopole invariants are both in the same class, which is associated to the Seiberg-Witten invariants (as the topological information that both give is encoded in the basic classes of the manifold). In order to explore this kind of behaviour, the first problem which should be addressed is the analysis of the different topological field theories which arise from the twisted $N = 2$ supersymmetric QCD. Conversely, one could check the predictions of the physical theories by comparing them to mathematical results. This would give a very fruitfull arena for the interaction of physics and mathematics which topological field theories have made possible.

Note added: During the review of this work we have rederived the bulk structure of the vacuum using non-perturbative $N = 1$ supersymmetric methods. We present this analysis in Appendix B. We have also included some details on the explicit realization of the ρ symmetry in Appendix A.

APPENDIX A

In this appendix we will make some observations concerning the parity symmetry (3.4). As this symmetry is a consequence of the isomorphism between the **2** representation and the $\bar{\mathbf{2}}$ representation of $SU(2)$, we will construct explicitly this isomorphism. We will focus on the case $N_f = 1$, although these considerations extend inmediately to the general case.

When $N_f = 1$, in $N = 1$ language we have a chiral superfield (quark) Q_a transforming in the **2** of $SU(2)$ and another chiral superfield (antiquark) $\widetilde{Q}_a$ transforming in the $\bar{\mathbf{2}}$. The index a is a color index. Now we can define the fields:

$$
\begin{aligned}
\hat{Q}_a^1 &= Q_a, \\
\hat{Q}_a^2 &= (\sigma_2)_{ab}\widetilde{Q}_b,
\end{aligned}
\tag{A.1}
$$

where σ_2 is a Pauli matrix. If $U \in SU(2)$, as $\sigma_2 U^* \sigma_2 = U$, the field $\hat{Q}_a^2$ transforms also in the **2**. This is an explicit realization of the isomorphism $\mathbf{2} \simeq \bar{\mathbf{2}}$. We must also redefine the chiral superfield Φ, which lives in the adjoint representation, in the following way:

$$
\hat{\Phi} = (\sigma_2)^{\mathrm{T}}\Phi.
$$

The new field $\hat{\Phi}$ is a symmetric matrix because $\mathrm{Tr}\Phi = 0$. The $N = 2$ coupling $\widetilde{Q}^{\mathrm{T}}\Phi Q$ is written in terms of the new variables as:

$$
\frac{1}{2}(\hat{Q}_a^2\hat{\Phi}_{ab}\hat{Q}_b^1 + \hat{Q}_a^1\hat{\Phi}_{ab}\hat{Q}_b^2).
\tag{A.2}
$$

The $N = 2$ mass term for the matter fields involves the gauge invariant quantity $X = \widetilde{Q}_a Q_a$, which in the new variables is written in the form,

$$
X = -i(\hat{Q}_2^2\hat{Q}_1^1 - \hat{Q}_2^1\hat{Q}_1^2).
\tag{A.3}
$$

The parity transformation, which interchanges the quark and the antiquark, must

be properly understood in terms of these variables as,

$$\rho : \hat{Q}^1 \leftrightarrow \hat{Q}^2. \tag{A.4}$$

As the term (A.2) is invariant under (A.4), this is a symmetry of $N = 2$ QCD with massless matter fields. Notice however that the $SU(2)$ singlet X changes its sign under (A.4), as it is obvious from (A.3). Therefore the $N = 2$ mass term for the quark and the antiquark changes its sign accordingly.

Another set of variables which is useful to take into account the ρ symmetry is the following:

$$\begin{aligned} Q^1 &= \frac{1}{2i}(\hat{Q}^1 - \hat{Q}^2), \\ Q^2 &= \frac{1}{2}(\hat{Q}^1 + \hat{Q}^2). \end{aligned} \tag{A.5}$$

The $N = 2$ coupling in these new variables reads as,

$$Q^1 \hat{\Phi} Q^1 + Q^2 \hat{\Phi} Q'$$

while the singlet X takes the form:

$$X = 2(Q_1^1 Q_2^2 - Q_2^1 Q_1^2).$$

The parity symmetry in terms of these variables is,

$$\begin{aligned} Q^1 &\to -Q^1, \\ Q^2 &\to Q^2. \end{aligned} \tag{A.6}$$

Using the variables defined in (A.5) it is easy to see that the flavour symmetry for the $N = 2$ QCD with gauge group $SU(2)$ and N_f hypermultiplets is $O(2N_f)$.

APPENDIX B

In this appendix we will derive the vacuum structure and the pattern of chiral symmetry breaking of the $N = 1$ theory with a mass term for the superfield Φ using the non-perturbative methods developed in [1-5]. This also shows that in certain conditions exact results for $N = 1$ theories can be useful in topological computations.

The $N = 1$ theory we are interested in is an $SU(2)$ theory with a quark Q, an antiquark $\widetilde{Q}$ and a triplet Φ. Apart from the minimal gauge couplings to the Yang-Mills field, this theory has a coupling between these three matter fields coming from the $N = 2$ supersymmetry (the last two terms in (2.33)) and also a mass term for Φ given in (2.45). The vacua of this theory can be found as the minima of the exact superpotential, and to obtain this we can use a technique developed in [4] and called the "integrating in" procedure. This technique allows one to obtain the exact superpotential for an "upstairs" theory starting from the one of a "downstairs" theory. The upstairs theory differs from the downstairs theory in that it contains an additional matter field. In our case we can take as the downstairs theory the $SU(2)$ theory with a quark and an antiquark, whose exact superpotential is known [32, 2], and as the additional field for the upstairs theory the chiral superfield in the adjoint representation, Φ. To "integrate in" the field Φ we must consider the gauge-invariant polynomials which include this field. In our case they are simply,

$$U = \text{Tr}\Phi^2, \quad Z = \sqrt{2}\widetilde{Q}\Phi, Q \tag{B.1}$$

and we must turn on a tree-level superpotential:

$$W_{\text{tree}} = mU + \lambda Z. \tag{B.2}$$

The scales Λ_d of the downstairs theory and Λ of the upstairs theory with the mass

term in (B.2) are related according to the principle of simple thresholds [4]:

$$\Lambda_d^5 = \Lambda^3 m^2. \tag{B.3}$$

The full superpotential of the upstairs theory with the additional tree-level term (B.2) is given by the principle of linearity [3, 4],

$$W_f(X, U, Z, \Lambda^3, m, \lambda) = W_u(X, U, Z, \Lambda^3) + mU + \lambda Z, \tag{B.4}$$

where X is the gauge-invariant polynomial of the downstairs theory, $X = \widetilde{Q}Q$, and W_u is the exact superpotential of the upstairs theory we are looking for. If we integrate out the field Φ and, correspondingly, the fields U and Z, we obtain a new superpotential:

$$W_l(X, \Lambda^3, m, \lambda) = W_d(X, \Lambda_d^5) + W_I(X, \Lambda^3, m, \lambda). \tag{B.5}$$

In this equation W_d is the dynamically generated superpotential of the downstairs theory and is given by,

$$W_d(X, \Lambda_d^5) = \frac{\Lambda_d^5}{X}. \tag{B.6}$$

In (B.5) W_I is an additional term which must be determined using the symmetries of the problem together with holomorphy principles and the behaviour of the superpotential in various limits. The first contribution to this piece comes after integrating out Φ from W_{tree}. In this case the result is [5],

$$W_{\text{tree,d}} = -\frac{\lambda^2}{4m} X^2.$$

The upstairs theory has two non-anomalous symmetries which can be used to constrain the form of W_I, following the methods of [1]. The first one is a $U(1)$ symmetry under which Q, $\widetilde{Q}$, Φ, m and λ have charges 2, 2, -1, 2 and -3, respectively. The other one is a $U(1)_{\mathcal{R}}$ symmetry with charges 1, 1, -1, 0 and -3.

The invariance of the superpotential under these symmetries and the holomorphy determine the form of W_I:

$$W_I = \frac{X^2 \lambda^2}{m} f\left(\frac{\Lambda^3 m^3}{X^3 \lambda^2}\right),$$

where $f(u) = \sum_{n=0}^{\infty} a_n u^n$ is an analytic function. Notice that the first term of this expansion corresponds to $W_{\text{tree,d}}$. Now, in the $m \to \infty$ limit, only W_d survives, and this implies that the coefficients a_n in the expansion of $f(u)$ must be zero for $n > 0$. Therefore $W_I = W_{\text{tree,d}}$ and the superpotential (B.5) is given by:

$$W_l(X, \Lambda^3, m, \lambda) = \frac{m^2 \Lambda^3}{X} - \frac{\lambda^2}{4m} X^2. \tag{B.7}$$

The interesting thing now is that W_l is the Legendre transform of W_u, which can be obtained from

$$W_n = W_l(X, \Lambda^3, m, \lambda) - mU - \lambda Z, \tag{B.8}$$

by integrating out m and λ, $i.e.$, by an inverse Legendre transform. The expectation values of these parameters are:

$$m = \frac{XU}{2\Lambda^3}\left(1 - \frac{Z^2}{X^2 U}\right), \quad \lambda = -\frac{ZU}{X\Lambda^3}\left(1 - \frac{Z^2}{X^2 U}\right), \tag{B.9}$$

and substituting these values in (B.8) one gets the superpotential of the upstairs theory [5]:

$$W_u = -\frac{XU^2}{4\Lambda^3}\left(1 - \frac{Z^2}{X^2 U}\right)^2. \tag{B.10}$$

Now we want to obtain the vacua of the $N = 2$ theory perturbed by the $N = 1$ mass term for Φ. Because of the principle of linearity, the superpotential of this

theory is given by (B.10) plus (B.2) with $\lambda = 1$ due to $N = 2$ supersymmetry:

$$W = -\frac{XU^2}{4\Lambda^3}\left(1 - \frac{Z^2}{X^2U}\right)^2 + mU + Z. \tag{B.11}$$

The equation $\partial W/\partial X = 0$ gives,

$$\frac{Z^2}{X^2U} = -\frac{1}{3},$$

which together with $\partial W/\partial Z = 0$ leads to,

$$U^3 = -\frac{27}{256}\Lambda^6. \tag{B.12}$$

This theory has therefore three vacua, corresponding to the three roots of this equation. This is in agreement with the results obtained from the $N = 2$ point of view. Finally, we have the vacuum expectation value for the field X in these vacua given by the roots of

$$X^3 = \frac{1}{2}m^3\Lambda^3. \tag{B.13}$$

This non-zero vacuum expectation value corresponds to the spontaneous breaking of the chiral symmetry in (3.10), as it happens in [32]. The subgroup of $\mathbf{Z}_{12}$ which preserves the vacuum expectation value for the guge invariant order parameter is precisely (3.11). In this way we have rederived all the results about the bulk structure of the vacua using non-perturbative methods for $N = 1$ theories.

53

REFERENCES

1. N. Seiberg, *Phys. Lett.* **B318**(1993), 469

2. N. Seiberg, *Phys. Rev.* **D49**(1994), 6857

3. K. Intriligator, R.G. Leigh and N. Seiberg, *Phys. Rev.* **D50**(1994), 1052

4. K. Intriligator, *Phys. Lett.* **B336**(1994), 409

5. K. Intriligator and N. Seiberg, *Nucl. Phys.* **B431**(1994), 551

6. N. Seiberg and E. Witten, *Nucl. Phys.* **B426**(1994), 19

7. N. Seiberg and E. Witten, *Nucl. Phys.* **B431**(1994), 484

8. E. Witten, *Math. Res. Lett.* **1**(1994), 769

9. S.K. Donaldson, *J. Diff. Geom.* **18**(1983), 279

10. S.K. Donaldson, *Topology* **29**(1990), 257

11. S.K. Donaldson and P.B. Kronheimer, *The Geometry Of Four-Manifolds*, Oxford Mathematical Monographs, 1990

12. E. Witten, *Comm. Math. Phys.* **117**(1988), 353

13. J.M.F. Labastida and M. Mariño, *Phys. Lett.* **B351**(1995), 146

14. E. Witten, *J. Math. Phys.* **35**(1994), 5101

15. E. Witten, *Nucl. Phys.* **B202**(1982), 253

16. C. Vafa and E. Witten, *Nucl. Phys.* **B431**(1994), 3

17. A. Karlhede and M. Roček, *Phys. Lett.* **B212**(1988), 51

18. M. Alvarez and J.M.F. Labastida, *Phys. Lett.* **B315**(1993), 251

19. M. Alvarez and J.M.F. Labastida, *Nucl. Phys.* **B437**(1995), 356

20. D. Anselmi and P. Fré, *Nucl. Phys.* **B392**(1993), 401, *Nucl. Phys.* **B404**(1993), 288, *Nucl. Phys.* **B416**(1994), 25, *Phys. Lett.* **B347**(1995), 247; M. Billó, R. D'Auria, S. Ferrara, P. Fré, P. Soriani and A. van Proeyen, "R-Symmetry and the Topological Twist of N=2 Effective Supergravities of Heterotic Strings", hep-th/9505123

21. J.M.F. Labastida and M. Mariño, *Nucl. Phys.* **B448**(1995), 373

22. S. Hyun, J. Park and J.S. Park, "Topological QCD", hep-th/9503201

23. Ch. Okonek and A. Teleman, "The Coupled Seiberg-Witten Equations, Vortices, And Moduli Spaces of Stable Pairs", to appear in *Int. J. Math.*

24. Ch. Okonek and A. Teleman, "Quaternionic Monopoles", alg-geom/9505029

25. S.B. Bradlow, *Comm. Math. Phys.* **135**(1990), 1, *J. Diff. Geom.* **33**(1991), 169

26. O. García-Prada, *Comm. Math. Phys.* **156**(1993), 527, *Int. J. Math.* **5**(1994), 1

27. S.K. Donaldson, *Proc. London Math. Soc.* **53**(1985), 1

28. N.J. Hitchin, *Proc. London Math. Soc.* **55**(1987), 59

29. O. García -Prada, J.M.F. Labastida and M. Mariño, to appear

30. S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, "Superspace", Benjamin, 1983

31. J.S. Park, *Comm. Math. Phys.* **163**(1994), 113, *Nucl. Phys.* **B423**(1994), 559

32. I. Affleck, M. Dine and N. Seiberg, *Nucl. Phys.* **B241**(1984), 493

33. E. Witten, "On S-Duality In Abelian Gauge Theories", hep-th/9505186

34. P.B. Kronheimer and T.S. Mrowka, *Bull. AMS* **30**(1994), 215